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Effective Lagrangians in bound state calculations

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Abstract

In order to investigate specific aspects of bound state calculations in a non-relativistic framework, we consider the energy-levels of a massive scalar particle, which moves in an external field and interacts in addition with a massless scalar particle. The discussion includes the following topics: dimensionally regularized bound-state calculations, ultraviolet finiteness of bound-state observables and their independence of the off-mass-shell behavior of Green functions, non-renormalizable interactions, structure of the non-relativistic two-point function, power counting and matching.

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I. INTRODUCTION

In several recent publications [1–3], the decay of $\pi^+\pi^-$ atoms has been studied in the framework of a non-relativistic effective Lagrangian - a method originally proposed by Caswell and Lepage [4] to investigate bound states in general. This method has proven to be far more efficient for the treatment of loosely bound systems - such as the $\pi^+\pi^-$ atom - than conventional approaches based on relativistic bound-state equations [5]. In [2], we have discussed the chiral expansion of the decay width in this framework. The corresponding numerical analysis is carried out in [3].

While performing that investigation, we have been confronted with the following specific features of this approach:

i) The non-relativistic effective Lagrangian defines a non-renormalizable Quantum Field Theory, where Green functions are expanded in powers of the coupling constant and of external momenta. While the expansion in the coupling constant can be done in the standard manner, the expansion in momenta is highly non-trivial: In order to make sense of the perturbative expansion, one needs a procedure where the contributions of higher dimensional operators in the loop expansion are suppressed. This problem goes under the name "power counting".

ii) The renormalization program in this non-relativistic theory must be carried out in a consistent manner. A reduction formula should be available, that allows one to evaluate scattering matrix elements from the residues of off-shell Green functions, analogous to the procedure used in the relativistic formulation. For this, the structure of the two-point function in the non-relativistic theory must be known.

iii) One needs a systematic framework to calculate energy-levels and decay widths in the non-relativistic theory. In view of applications in hadronic atom calculations, the framework should allow one to also evaluate the lifetime of unstable states in an unambiguous manner.

iv) The effective Lagrangian contains terms with arbitrary high powers of space derivatives. As a result of this, the matrix elements of the perturbation Hamiltonian between the unperturbed wave functions start to diverge at a sufficiently high order of the perturbative expansion. One should demonstrate that these divergences cancel at any given order in the expansion.

v) The scattering sector of $\pi\pi$ interactions is most effectively described by Chiral Perturbation Theory (ChPT). Here, the ultraviolet (UV) divergences are removed by higher order counterterms in the chiral expansion. One needs an analogous statement concerning the ultraviolet divergences in bound-state observables.

Some of these issues - like the problem of power counting in the context of QED bound states [6–8], NRQCD and HQET [9–12], effective field theories in the one [13–15] and two-nucleon sectors [16] - have been treated previously in the literature, while some others have not yet been, as far as we are aware, considered in detail. In particular, a detailed discussion of the renormalization program in the non-relativistic sector - including an investigation of

the structure of the two-point function - is not yet available to the best of our knowledge. For this reason, we present in this article two examples of Quantum Field Theories, where these issues can be studied in an unambiguous manner.

The article is organized as follows. In section II, we present the relativistic formulation of the models: First, we consider a massive scalar particle moving in a given external field. Second, we add a Yukawa interaction of the heavy particle with a massless scalar particle. This interaction changes the energy-levels - we evaluate the shift at lowest non-trivial order in the ground state. This section also contains an example of a non-renormalizable interaction, that nevertheless results in ultraviolet finite shifts of the energy-levels. In section III, we formulate the non-relativistic version of the external field problem. A general formula for evaluating bound-state poles is derived with the use of Feshbach's [17] projection technique. We determine the perturbative expansion of the energy-levels, using dimensional regularization to tame the ultraviolet divergences that occur in these calculations. In section IV, we discuss the inclusion of a dynamical light field in the non-relativistic framework, and illustrate the problem generated by loop corrections. The section V is devoted to a detailed study of the two-point function in the non-relativistic theory, while the matching procedure, needed to determine the coefficients in the non-relativistic Lagrangian, is considered in section VI. In section VII, we evaluate the energy-level shift in the ground state and explicitly verify that the result agrees with the relativistic calculation performed in section II. Finally, section VIII contains a summary and concluding remarks. The Appendices are devoted to technical details of the calculations presented in the text.

II. RELATIVISTIC FORMULATION

We investigate in this section the bound state spectrum of a heavy particle, which is bound in an external Coulomb field, and which interacts in addition with a massless scalar particle. The results of the relativistic formulation outlined here are compared in later sections with a non-relativistic framework.

A. The external field

We consider a heavy, complex scalar field H of mass M, interacting with an external, static source ϕ,

$$\mathcal{L} = \partial_\mu H^\dagger \partial^\mu H - M^2 H^\dagger H + \lambda H^\dagger H \phi \; . \tag{2.1}$$

To determine the energy spectrum, one may proceed in several ways. In the following, we consider the connected two-point function

$$\bar{G}(x,y) = i\langle 0|T H(x) H^\dagger(y)|0\rangle_c \;\; , \tag{2.2}$$

because its Fourier transform contains the bound state poles. It is useful to note that $\bar{G}$ is the inverse of a differential operator,

$$(D_M - \lambda\phi)\bar{G}(x,y) = \delta^4(x-y) \;\; ; \;\; D_M = \Box_x + M^2 \;\; , \tag{2.3}$$

as a result of which one has[1]

$$\bar{G} = D_M^{-1} + D_M^{-1} T D_M^{-1} \ ,$$
$$T = (1 - \lambda\phi D_M^{-1})^{-1}\lambda\phi = \lambda\phi + \lambda^2\phi D_M^{-1}\phi + O(\lambda^3) \ . \tag{2.4}$$

The scattering amplitude is obtained from the Fourier transform of T by putting the momenta on the mass-shell,

$$\int dx^0 d^3\mathbf{x} d^3\mathbf{y} e^{ip^0(x^0-y^0)} e^{-i\mathbf{px}} e^{i\mathbf{ky}} T(x,y) = T(\mathbf{p},\mathbf{k}) \ , \tag{2.5}$$

with

$$p^2 = M^2 \ , \ |\mathbf{p}| = |\mathbf{k}| \ . \tag{2.6}$$

Expanding in powers of λ, we find

$$T(\mathbf{p},\mathbf{k}) = \sum_{n=1}^{\infty} \lambda^n \int K_n(\mathbf{p},\mathbf{k};\mathbf{q}_1,\ldots,\mathbf{q}_n) \prod_{i=1}^{n} \phi(\mathbf{q}_i) d\nu(\mathbf{q}_i) \ , \tag{2.7}$$

with

$$\phi(\mathbf{q}) = \int d^3\mathbf{x} e^{-i\mathbf{qx}} \phi(\mathbf{x}) \, , \qquad d\nu(\mathbf{q}) = d^3\mathbf{q}/(2\pi)^3 \ . \tag{2.8}$$

The kernels K_n are independent of the mass M,

$$K_1(\mathbf{p},\mathbf{k};\mathbf{q}_1) = (2\pi)^3 \delta^3(\mathbf{q}_1 - \mathbf{p} + \mathbf{k}) \, ,$$
$$K_2(\mathbf{p},\mathbf{k};\mathbf{q}_1,\mathbf{q}_2) = (2\pi)^3 \delta^3(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{p} + \mathbf{k}) \frac{1}{(\mathbf{k}+\mathbf{q}_2)^2 - \mathbf{k}^2} \ . \tag{2.9}$$

In the following, we often consider the case where the external field is of the Coulomb-type,

$$\phi = \frac{1}{|\mathbf{x}|} \, . \tag{2.10}$$

Below, we shall not always distinguish between the case of short and long range (Coulomb) external sources. In the latter case, the on-shell scattering amplitude contains infrared divergences that are tamed by using dimensional regularization.

The Fourier transform of the propagator

$$\bar{G}(E;\mathbf{p},\mathbf{k}) = \int dx^0 d^3\mathbf{x} d^3\mathbf{y} e^{iE(x^0-y^0)} e^{-i\mathbf{px}} e^{i\mathbf{ky}} \bar{G}(x,y) \tag{2.11}$$

is then closely related to Schwinger's Green function [18],

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[1] We use throughout the Landau symbols $O(x)$ [$o(x)$] for quantities that vanish like x [faster than x] when x tends to zero. Furthermore, it is understood that this holds modulo logarithmic terms, i.e. we write also $O(x)$ for $x \ln x$.

$$\bar{G}(E;\mathbf{p},\mathbf{k}) = -\frac{(2\pi)^3}{2M}\, G_S\Big(\frac{E^2-M^2}{2M};\mathbf{p},\mathbf{k}\Big)_{Ze^2=\frac{\lambda}{2M}} \,. \tag{2.12}$$

The explicit expression is given by[2]

$$\bar{G}(E;\mathbf{p},\mathbf{k}) = -\frac{(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})}{E^2-M^2-\mathbf{p}^2} + \frac{1}{E^2-M^2-\mathbf{p}^2}\,\frac{4\pi\lambda}{|\mathbf{p}-\mathbf{k}|^2}\,\frac{1}{E^2-M^2-\mathbf{k}^2}$$
$$+\,\frac{1}{E^2-M^2-\mathbf{p}^2}\;4\pi\lambda\nu I(E;\mathbf{p},\mathbf{k})\;\frac{1}{E^2-M^2-\mathbf{k}^2}\,, \tag{2.13}$$

with

$$I(E;\mathbf{p},\mathbf{k}) = \int_0^1 \frac{\rho^{-\nu}d\rho}{[(\mathbf{p}-\mathbf{k})^2\rho+\nu^2/\lambda^2(1-\rho)^2(M^2+\mathbf{p}^2-E^2)(M^2+\mathbf{k}^2-E^2)]} \;\;, \tag{2.14}$$

where $\nu = \frac{1}{2}\,\lambda\,(M^2-E^2)^{-1/2}$.

In the representation (2.13), the two-point function is split into three parts. The two first terms correspond to free propagation and to one interaction with the external field, respectively, whereas the third term describes multiple interactions. The function $\bar{G}(E;\mathbf{p},\mathbf{k})$ develops poles at $\nu = n$ or, equivalently, at

$$E_n^2 = M^2 - \frac{\lambda^2}{4n^2}\,. \tag{2.15}$$

The residues factorize in a product of two wave functions which are solutions of the Schrödinger equation with a Coulomb potential,

$$\Big[E_n^2 - M^2 - \mathbf{p}^2\Big]\,\psi_n(\mathbf{p}) = -4\pi\lambda\int\frac{d\nu(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^2}\psi_n(\mathbf{q})\,. \tag{2.16}$$

For later use, we display the normalized ground-state wave function in momentum space,

$$\psi_1(\mathbf{p}) = \frac{(64\pi\gamma^5)^{1/2}}{(\mathbf{p}^2+\gamma^2)^2}\;\;,\gamma=\lambda/2\;\;;\qquad \int d\nu(\mathbf{p})\,|\psi_1(\mathbf{p})|^2 = 1\,. \tag{2.17}$$

B. Radiative corrections

We now supplement the Lagrangian (2.1) with a Yukawa-interaction of the heavy particle with a massless scalar field. This additional interaction affects the energy-levels (2.15) - we shall investigate the induced change in E_1^2 at lowest non-trivial order in the Yukawa coupling.

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[2]In order to simplify the notation, we omit the negative imaginary part in M^2 that is understood here and in the following.

1. The Lagrangian

The Lagrangian is

$$\mathcal{L} = \partial^\mu H^\dagger \partial_\mu H - M^2 H^\dagger H + \frac{1}{2}(\partial^\mu \ell \partial_\mu \ell - m^2 \ell^2) + H^\dagger H(\lambda\phi + e\ell) + \mathcal{L}_{c.t.} \ ,$$
$$\mathcal{L}_{c.t.} = -\delta M^2 H^\dagger H - \frac{1}{2}\delta m^2 \ell^2 + (c_0 + c_1\lambda\phi + c_2\lambda^2\phi^2)\ell \ . \tag{2.18}$$

The mass m is used as an infrared regulator in intermediate steps. Whenever possible, we always consider the limiting case $m \to 0$. The counterterms - collected in the quantity $\mathcal{L}_{c.t.}$ - cancel the ultraviolet divergences in the diagrams displayed in Fig. 1. As is discussed below, the counterterm proportional to c_2 is needed in order to have a smooth behavior of the scattering amplitude in the limit $m \to 0$.

In dimensional regularization, we find

$$\delta M^2 = -e^2 \bar{L}_M + \frac{e^2}{8\pi^2} + O\Big(\frac{m}{M}\Big) \ , \quad \delta m^2 = -e^2 \bar{L}_M + O\Big(\frac{m^2}{M^2}\Big) \ ,$$
$$c_0 = -eM^2 \bar{L}_M + \frac{eM^2}{16\pi^2} \ , \quad c_1 = e\bar{L}_M \ , \quad c_2 = -\frac{e}{32\pi^2 M^2} \ , \tag{2.19}$$

where D stands for the dimension of the space-time, and

$$\bar{L}_M = \frac{1}{8\pi^2}\Big(\bar{L}(\mu) + \frac{1}{2}\ln\frac{M^2}{\mu^2}\Big) \ ,$$
$$\bar{L}(\mu) = \mu^{D-4}\left(\frac{1}{D-4} - \frac{1}{2}(\Gamma'(1) + \ln 4\pi)\right), \tag{2.20}$$

where μ denotes the renormalization scale. The finite parts of the mass counterterms are chosen such that M coincides with the physical mass of the heavy particle at order e^2, and that the light field stays massless at this order. As is discussed below, the finite parts of the couplings c_i as chosen in (2.19) guarantee a smooth behavior of the scattering amplitude in the limit $m \to 0$. With (2.19), the Green functions of the heavy and light fields are ultraviolet finite at order e^2 and at any order in λ.

2. Perturbation theory for bound states

The Yukawa interaction (2.18) generates a shift in the energy-levels (2.15). In order to derive the formula that allows us to calculate this shift, we start from the connected two-point function of the heavy scalar field in (2.18),

$$G(x,y) = i\langle 0|TH(x)H^\dagger(y)|0\rangle_c \ . \tag{2.21}$$

It obeys the familiar Dyson-Schwinger equation

$$G(x,y) = \bar{G}(x,y) + \int dudv\, \bar{G}(x,u)\Sigma(u,v)G(v,y) \ , \tag{2.22}$$

where $\bar{G}$ stands for the Green function (2.2) in the absence of the Yukawa interaction. The relation (2.22) defines the self-energy Σ of the heavy scalar particle. The expansion of Σ in powers of the coupling constant e takes the form

$$\Sigma(u,v) = \Sigma_2(u,v) + \Sigma_4(u,v) + \cdots \ , \tag{2.23}$$

where $\Sigma_n = O(e^n)$. Explicitly, we find for the lowest order expression

$$\begin{aligned}\Sigma_2(u,v) = e\delta(u-v)\int dw\, D_m^{-1}(u-w)\,[-ie\bar{G}(w,w) + c_0 + c_1\lambda\phi(w) + c_2\lambda^2\phi^2(w)] \\ - \delta M^2\delta(u-v) - ie^2\, D_m^{-1}(u-v)\bar{G}(u,v) \ \ ,\end{aligned} \tag{2.24}$$

where $D_m^{-1}(x)$ denotes the propagator of the free two-point function of the light field, $(\Box + m^2)D_m^{-1}(x) = \delta^4(x)$.

It is convenient to define the Fourier transforms

$$\Sigma(E;\mathbf{x},\mathbf{y}) = \int dx^0 e^{iE(x^0-y^0)}\,\Sigma(x,y) \ \ , \tag{2.25}$$

and

$$\Sigma(E;\mathbf{p},\mathbf{k}) = \int d^3\mathbf{x}d^3\mathbf{y}\, e^{-i\mathbf{p}\mathbf{x}+i\mathbf{k}\mathbf{y}}\Sigma(E;\mathbf{x},\mathbf{y}) \ . \tag{2.26}$$

The diagrams contributing to Σ_2 are depicted in Fig. 2. We have not displayed the contributions generated by the counterterms c_i and δM^2. There are two types of diagrams: the tadpole-like diagram I, where the loop is generated by the propagator $\bar{G}$, and the self-energy diagram II, generated by a closed loop with the light scalar and $\bar{G}$. The tadpole-like diagram leads to a self-energy of the form

$$\Sigma_2^I(E;\mathbf{p},\mathbf{k}) = e^2\frac{R(\mathbf{q})}{m^2+\mathbf{q}^2} \ \ ; \ \mathbf{q} = \mathbf{p}-\mathbf{k} \ \ , \tag{2.27}$$

where the residue behaves at low momentum transfer as

$$R = R_0\delta^3(\mathbf{q}) + \frac{R_1}{\mathbf{q}^2} + \frac{R_2}{|\mathbf{q}|} + R_3\ln\frac{|\mathbf{q}|}{M} + O(1) \ \ , \ \mathbf{q}\to 0 \ \ , \tag{2.28}$$

and where $R_m = O(\lambda^m)$. The finite parts in the counterterms are chosen such that $R_{0,1,2} = 0$. As a result of this, the regulator mass m can be sent to zero, and the self-energy is not more singular (except from logarithms) than the external Coulomb-field in the forward region $\mathbf{p} = \mathbf{k}$. In particular, one has

$$\Sigma_2^I(E;\mathbf{p},\mathbf{k}) = -\frac{\lambda e^2}{24\pi M^2}\frac{1}{\mathbf{q}^2}\left\{1 + O(\mathbf{q}^2)\right\} + O(e^2\lambda^2) \ . \tag{2.29}$$

The coupling to the external Coulomb-field is modified accordingly at this order,

$$\lambda \to \lambda\left[1 - \frac{e^2}{96\pi^2M^2}\right] \ . \tag{2.30}$$

Note that in QED, the coefficients $R_{0,1,2}$ are identically zero by charge conjugation and gauge invariance.

Coming back to the bound-state energy, we observe that the Fourier transform of $G(x,y)$ has a pole at the energy E_B corresponding to the bound state B,

$$G(E;\mathbf{x},\mathbf{y}) = \int dx^0 e^{iE(x^0-y^0)}\, G(x,y) \to -\frac{\langle 0|H(0,\mathbf{x})|B\rangle\langle B|H^\dagger(0,\mathbf{y})|0\rangle}{2E_B(E-E_B)} + \cdots \,. \tag{2.31}$$

Using Eqs. (2.3), (2.22) and (2.31), one obtains for the bound-state wave function, defined as $\psi(\mathbf{x}) = \langle 0|H(0,\mathbf{x})|B\rangle$, the equation

$$(\, E_B^2 - M^2 + \triangle_x + \lambda\phi(\mathbf{x})\,)\,\psi(\mathbf{x}) = -\int d^3\mathbf{y}\,\Sigma(E_B;\mathbf{x},\mathbf{y})\,\psi(\mathbf{y})\,. \tag{2.32}$$

In order to calculate the corrections to the energy-levels, one can apply ordinary perturbation theory to the Schrödinger-like equation (2.32). The result is

$$\Delta {E_n}^2 = \bar{E}_n^2 - E_n^2 = -\int d\nu(\mathbf{p})\, d\nu(\mathbf{k})\ \psi_n^*(\mathbf{p})\,\Sigma_2(E_n;\mathbf{p},\mathbf{k})\,\psi_n(\mathbf{k}) + O(e^4)\,. \tag{2.33}$$

Here ψ_n denotes the normalized unperturbed Coulomb eigenfunction (2.16), whereas $\bar{E}_n^2$ and E_n^2 stand for the true and unperturbed eigenvalues, respectively [19].

3. The ground state

We are now ready to evaluate the shift in the energy-levels, generated by the Yukawa interaction $eH^\dagger Hl$ in (2.18). We restrict ourselves to the calculation of the shift in the ground-state. By dimensional reasons, one has

$$\Delta E_1^2 = e^2 F(z) + O(e^4)\,, \qquad z = \lambda/M\,. \tag{2.34}$$

Below, we determine the leading term in the expansion of F in powers of z. The calculation can be performed at zero mass of the light particle, because the bound particle is always slightly off-shell - there is no additional infrared regulator needed. The shift is naturally split into the two parts originating from the diagrams of type I and II displayed in Fig. 2. We split the contributions from the diagrams of type II furthermore into a zero-Coulomb, a one-Coulomb and a multi-Coulomb part according to Eq. (2.13), and write

$$F = F_I + F_{II,0} + F_{II,1} + F_{II,m}\,. \tag{2.35}$$

The contribution from the diagrams of the type I is remarkably simple to evaluate. Indeed, the previous discussion has shown that - at leading order in λ - the inclusion of these diagrams merely amounts to an $O(e^2)$ correction in the strength of the Coulomb potential, see Eq. (2.30). The contributions of order λ^2 and higher in Σ_2^I may be discarded at leading order. One therefore has

$$F_I = \frac{z^2}{192\pi^2} + o(z^2)\,. \tag{2.36}$$

The expression for the zero-Coulomb part in the diagram II starts again at order z^2, modulated with a logarithmic term,

$$F_{II,0} = -\frac{z^2}{32\pi^2}\left(\ln z^2 - \frac{1}{2}\right) + o(z^2)\,. \tag{2.37}$$

The one-Coulomb contribution is given by

$$F_{II,1} = -\int d\nu(\mathbf{p}) d\nu(\mathbf{k})\ \psi_1^*(\mathbf{p})\ \frac{4\pi\lambda}{|\mathbf{p}-\mathbf{k}|^2}\ \Gamma(E_1;\mathbf{p},\mathbf{k})\ \psi_1(\mathbf{k})\ \ , \tag{2.38}$$

where Γ stands for the one-loop correction to the coupling of the heavy particle to the external field,

$$\Gamma(E_1;\mathbf{p},\mathbf{k}) = -\int \frac{d^4q}{(2\pi)^4 i}\ \frac{1}{q^2}\ \frac{1}{(M^2-(E_1-q^0)^2+(\mathbf{p}-\mathbf{q})^2)(M^2-(E_1-q^0)^2+(\mathbf{k}-\mathbf{q})^2)}\,. \tag{2.39}$$

The calculation of the one-Coulomb contribution to the energy shift is relegated to Appendix A. We simply quote the final result,

$$F_{II,1} = \frac{z^2}{32\pi^2}\left(\ln z^2 - 6 + 8\ln 2\right) + o(z^2)\,. \tag{2.40}$$

The logarithmic terms cancel in the sum of the zero- and one-Coulomb contributions.

The calculation of the multi-Coulomb contribution is given in Appendix B. The result can be written as

$$F_{II,m} = -\frac{z^2}{32\pi^2}\ J + o(z^2)\ \ , \tag{2.41}$$

with

$$J = 256\pi^2 \int_0^\infty \frac{x dx}{(1+x)^{1/2}} \int d\nu(\mathbf{u}) d\nu(\mathbf{v}) \int_0^1 \frac{d\rho \rho^{-(1+x)^{-1/2}}}{(1+\mathbf{u}^2)^2(1+\mathbf{v}^2)^2(1+x+\mathbf{u}^2)(1+x+\mathbf{v}^2)} \times$$
$$\times \frac{1}{(\mathbf{u}-\mathbf{v})^2\rho + (1-\rho)^2(1+x+\mathbf{u}^2)(1+x+\mathbf{v}^2)/(4(1+x))}\,. \tag{2.42}$$

Collecting everything, we find

$$F(z) = -\frac{z^2}{32\pi^2}\left(\frac{16}{3} - 8\ln 2 + J\right) + o(z^2)\,. \tag{2.43}$$

C. Non-renormalizable interactions

According to Eq. (2.33), the corrections to the Coulomb energy-spectrum are obtained at leading order by folding the self-energy with the unperturbed Coulomb wave functions in momentum space. In theories with non-renormalizable interactions - e.g. in ChPT - the

amplitudes grow faster in the external momenta at every order in the loop expansion, and the just mentioned integration starts to diverge at some order. This divergence is not, at first glance, cured by the counterterms in the Lagrangian, because it arises from an additional integration of the matrix element that was already made finite by these counterterms. In the context of the $\pi^+\pi^-$-atom, it has been claimed in Ref. [20] that, for this reason, one needs to introduce an explicit ultraviolet cutoff in order to render the bound-state observables finite. Here we wish to demonstrate in a simple model - bearing all the important properties of the original problem - that this is not the case: the ultraviolet divergences that occur at intermediate steps in the evaluation of the perturbed energy-levels are all cancelled.

In order to illustrate the point, we consider the Lagrangian

$$\mathcal{L} = \partial_\mu H^\dagger \partial^\mu H - M^2 H^\dagger H + \lambda H^\dagger H \phi + g \left\{ (\Box + M^2) H^\dagger \right\} \lambda \phi \left\{ (\Box + M^2) H \right\} + \mathcal{L}_{c.t.} \,, \qquad (2.44)$$

where ϕ is a Coulomb field. In the last term, we anticipate the necessity of adding counterterms to cancel the ultraviolet divergences in the Green functions.

The self-energy part is again defined by (2.22), and the shift in the energy-levels can be calculated from (2.33), with $e^2 \to g$. At order g, we find for the self-energy part

$$\Sigma(E; \mathbf{p}, \mathbf{k}) = \, (M^2 - E^2 + \mathbf{p}^2) \, \frac{4\pi\lambda g}{|\mathbf{p} - \mathbf{k}|^2} \, (M^2 - E^2 + \mathbf{k}^2) + \Sigma_{c.t.}(E; \mathbf{p}, \mathbf{k}) \, . \qquad (2.45)$$

The first term is ultraviolet finite - however, it leads to an ultraviolet infinite contribution to the energy shift at order g. This fact mimics the situation which one encounters in the calculation of the energy of hadronic atoms beyond the so-called "local" approximation [20]. The remedy for the difficulty is provided by the counterterm in Eq. (2.44): as one may observe from Eq. (2.22), the two-point function [defined with the Lagrangian (2.44)] is infinite at order g, although the self-energy is not. In dimensional regularization, the ultraviolet divergent piece in the two-point function is given by

$$G(E; \mathbf{p}, \mathbf{k})\Big|_{div} = -2\pi\lambda^3 g L(\mu) \int d\nu(\mathbf{q}) d\nu(\mathbf{l}) \bar{G}(E; \mathbf{p}, \mathbf{q}) \; \bar{G}(E; \mathbf{l}, \mathbf{k}) + O(g^2) \, , \qquad (2.46)$$

where $L(\mu)$ is given by

$$L(\mu) = (\mu^2)^{D-4} \Big(\frac{1}{D-4} - \Gamma'(1) - \ln 4\pi \Big) \, . \qquad (2.47)$$

This divergence is cancelled by the counterterm

$$\mathcal{L}_{c.t.} = -\frac{1}{2} \, \lambda^3 g \, c \, H^\dagger(x) H(x) \triangle \phi(\mathbf{x}); \qquad c = L(\mu) + c^r(\mu) \;\; , \qquad (2.48)$$

where $c^r(\mu)$ denotes the arbitrary scale-dependent finite part of c. If one would carry out the perturbative expansion of the two-point function to higher orders in g, one would have to introduce additional counterterms, containing higher derivatives - the situation clearly resembles the one in ChPT. Evaluating $\Sigma_{c.t.}$ generated by the counterterm (2.48), one can verify that the corresponding contribution to the energy-level shift cancels the ultraviolet divergence generated by the first term in (2.45), leading to a finite and scale-independent result at order g.

Let us note that the ultraviolet divergences can be removed from the bound-state observables in a perturbative manner: indeed, the divergent piece of the two-point function (2.46) was calculated at $O(g)$. The corresponding counterterm removes the ultraviolet divergence in the energy-levels at $O(g)$. This suggest that also in non-renormalizable theories, bound-state observables are rendered finite at a given order, provided that the renormalization in the scattering sector is carried out at the same order.

III. NON-RELATIVISTIC FORMULATION: THE EXTERNAL FIELD

In the remaining part of this article, we wish to reformulate the relativistic problem considered in section II in a non-relativistic framework. The energy-levels can then be calculated using Rayleigh-Schrödinger perturbation theory. At the same time, the formalism developed allows one to calculate the width of decaying states in a crystal clear manner, without using handwaving arguments. The procedure goes in two steps [4]. First, one constructs an effective theory that allows one to evaluate the scattering matrix elements in the low-energy region. Second, one uses the Hamiltonian of this effective theory to evaluate the bound state poles.

We restrict the considerations in this section to the non-relativistic formulation of the external field problem (2.1). Radiative corrections generated by the Yukawa interaction (2.18) will be treated afterwards.

A. The scattering sector

For the construction of the non-relativistic effective Lagrangian that is relevant for the theory defined in Eq. (2.1), one may proceed in several ways. Below, we choose a method that is based on the following observation. In the relativistic framework, the scattering matrix elements are obtained from the residue of the poles in the Fourier transform of the two-point function $\bar{G}$. These poles are generated, in the present case, by the free propagators,

$$D_M^{-1}(p) = \frac{1}{M^2 - p^2} = \frac{1}{2\omega_{\mathbf{p}}} \left(\frac{1}{\omega_{\mathbf{p}} - p^0} + \frac{1}{\omega_{\mathbf{p}} + p^0} \right) \;\; ; \; \omega_{\mathbf{p}} = (M^2 + \mathbf{p}^2)^{1/2} \;\; , \tag{3.1}$$

which have poles at $p^0 = \pm\omega_{\mathbf{p}}$. It is therefore sufficient to construct a non-relativistic theory that has the same structure in the vicinity of $p^0 = \omega_{\mathbf{p}}$. The pole terms in (3.1) can be generated with the inverse of the differential operators

$$\begin{aligned} D_\pm &= \pm i\partial_t - \sqrt{M^2 - \triangle} \;\; , \qquad D_\pm(x) \triangle_\pm(x) = \delta^{(4)}(x) \;\; , \\ \triangle_\pm(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{\pm p^0 - \sqrt{M^2 + \mathbf{p}^2}} \, . \end{aligned} \tag{3.2}$$

The Lagrangian will furthermore be quadratic in the non-relativistic heavy field h. Writing $\mathcal{L}_{NR} = h^\dagger \mathcal{D} h$, the differential operator must $\mathcal{D}$ contain, in addition to D_+, further terms that generate the transition amplitude T in (2.4). Writing

$$\mathcal{D} = D_+ + X \ , \qquad \mathcal{D}^{-1} = D_+^{-1} - D_+^{-1} T_{NR} D_+^{-1} \tag{3.3}$$

and requiring that T_{NR} is closely related to the relativistic scattering amplitude T in (2.4), one finds [21] that one may take

$$X = \lambda d\phi d(1 + \lambda D_-^{-1} d\phi d)^{-1} \ ; \qquad d \doteq (2\sqrt{M^2 - \triangle})^{-1/2} \ , \tag{3.4}$$

with

$$T_{NR} = dTd \ . \tag{3.5}$$

This formulation involves a non-local Lagrangian that generates the same truncated off-shell two-point function as the relativistic theory. In order to arrive at a local Lagrangian, one may expand the non-local operators $D_\pm^{-1}$ and d in inverse powers of the heavy mass M. In this manner, one arrives at a local Lagrangian, that contains, however, an infinite string of terms. Furthermore, it contains time derivative of arbitrary high orders, acting on the heavy field. These time derivatives may be eliminated in a standard manner by use of the equation of motion (EOM). We find that this can be shown in a transparent manner by use of external sources coupled to the heavy field. We relegate these manipulations to Appendix C and simply quote the result,

$$\begin{aligned}
\mathcal{L}_{NR} &= h^\dagger D_0 h + \sum_{n=0}^{\infty} \frac{1}{(2M)^{2n+1}} \mathcal{L}_{2n+1} \ , \\
D_0 &= i\partial_t - M + \triangle/2M \ , \\
\mathcal{L}_1 &= h^\dagger \lambda \phi h \ , \quad \mathcal{L}_3 = h^\dagger \left[\triangle^2 + \lambda(\phi\triangle + \triangle\phi) + \lambda^2\phi^2 \right] h \ , \\
\mathcal{L}_5 &= \frac{1}{2} h^\dagger \left[4\triangle^3 + \lambda \left(2\triangle\phi\triangle + 5\phi\triangle^2 + 5\triangle^2\phi \right) \right] h \\
&\quad + h^\dagger \left[\lambda^2 \left(2\phi^2\triangle + \triangle\phi^2 + 3\phi\triangle\phi \right) + 2\lambda^3\phi^3 \right] h \ .
\end{aligned} \tag{3.6}$$

The scattering matrix elements are obtained by evaluating the two-point function

$$G_{NR}(x,y) = i\langle 0|Th(x)h^\dagger(y)|0\rangle \tag{3.7}$$

in the presence of the interactions $\mathcal{L}_i$. The perturbation theory is performed as follows. First one sums up all mass insertions in the external lines. The Fourier transform of the two-point function then develops poles at $p^0 = \omega_{\mathbf{p}}$ and at $p^0 = \omega_{\mathbf{k}}$,

$$\begin{aligned}
G_{NR}(p^0; \mathbf{p}, \mathbf{k}) &= \int dx^0 d^3\mathbf{x} d^3\mathbf{y} e^{ip^0(x^0 - y^0)} e^{-i\mathbf{px}} e^{i\mathbf{ky}} \, G_{NR}(x,y) \\
&= \frac{(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k})}{\omega_p - p^0} + \frac{1}{\omega_{\mathbf{p}} - p^0} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \, T_{NR}(\mathbf{p}, \mathbf{k}; p^0) \, \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \frac{1}{\omega_{\mathbf{k}} - p^0} \ .
\end{aligned} \tag{3.8}$$

With the normalization chosen, T_{NR} coincides with the low-energy expansion of the relativistic amplitude T in (2.5) order by order. The presence of the factors $\sqrt{2\omega_{\mathbf{p}}}$, $\sqrt{2\omega_{\mathbf{k}}}$ in the above formula reflects the relation (3.5). The scattering amplitude is obtained by putting

the momenta on the mass-shell $p^0 = \omega_{\mathbf{p}}$, $|\mathbf{p}| = |\mathbf{k}|$. Expanding in powers of the coupling constant, one has

$$T_{NR}(\mathbf{p},\mathbf{k}) = \sum_{n=1}^{\infty} \lambda^n \int k_n(\mathbf{p},\mathbf{k};\mathbf{q}_1,\ldots,\mathbf{q}_n;M) \prod_{i=1}^{n} \phi(\mathbf{q}_i) d\nu(\mathbf{q}_i) \; . \tag{3.9}$$

In the internal lines, one need not fully sum the mass insertions. Instead, one sums, at a fixed order in λ, all terms at a given order in $1/M$,

$$k_n(\mathbf{p},\mathbf{k};\mathbf{q}_1,\ldots,\mathbf{q}_n;M) = \sum_{\nu=0}^{\infty} \frac{1}{M^\nu} k_{n,\nu}(\mathbf{p},\mathbf{k};\mathbf{q}_1,\ldots,\mathbf{q}_n) \; . \tag{3.10}$$

In Fig. 3 we display the corresponding graphs at order λ^2. The diagram Fig. 3A stands for the Green function evaluated in the relativistic theory, whereas mass insertions and the contact term in the non-relativistic theory are displayed in graphs Fig. 3B and Fig. 3C. The mass dependence cancels, order by order, and the leading term coincides with the matrix element in (2.9), evaluated in the relativistic theory,

$$k_{n,0} = K_n \;\; ; \;\; k_{n,i} = 0 \;\; , i \geq 1 \; . \tag{3.11}$$

We illustrate the procedure with the matrix element at order λ. The first two terms in the $1/M$ expansion give

$$T_{NR}(\mathbf{p},\mathbf{k}) = \lambda \phi(\mathbf{p}-\mathbf{k}) \frac{\omega_{\mathbf{p}}}{M} \left[1 - \frac{\mathbf{p}^2}{2M^2} + O(M^{-3}) \right] + O(\lambda^2) \; . \tag{3.12}$$

A different Lagrangian would be generated by use of the standard Foldy-Wouthuysen-Tani transformation, which gives

$$\mathcal{L}_{FWT} = h^\dagger (i\partial_t - \left[M^2 - \triangle - \lambda\phi \right]^{1/2}) h \; . \tag{3.13}$$

Again, the Lagrangian may be expanded in inverse powers of M. At order $1/M^5$ and higher, it differs from the one considered above,

$$\mathcal{L}_{NR} - \mathcal{L}_{FWT} = \frac{\lambda}{64M^5} h^\dagger \left\{ \triangle^2\phi - 2\triangle\phi\triangle + \phi\triangle^2 - 2\lambda\triangle\phi^2 + 2\lambda\phi\triangle\phi \right\} h + \cdots \; . \tag{3.14}$$

On the other hand, evaluating the scattering amplitude with (3.13), one finds that it agrees with the previous one on the mass-shell, to all orders in the coupling constant. The reason is the fact that the two Lagrangians may be transformed into each other by a field redefinition. To illustrate, the terms at order $1/M^5$ are removed if in $\mathcal{L}_{NR}$ we substitute

$$\begin{aligned} h &\to (1+X)h \, , \\ X &= -\frac{1}{32M^4} \left[\lambda(\triangle\phi - \phi\triangle) - \lambda^2\phi^2 \right] \; . \end{aligned} \tag{3.15}$$

On the other hand, the two-point functions differ off the mass-shell. Indeed, consider the term linear in λ. From (3.14) it follows that the difference is proportional to $\lambda\phi(\mathbf{p}-\mathbf{k})(\mathbf{p}^2-\mathbf{k}^2)^2$ at order M^{-5}.

B. Perturbation theory for bound states

In Quantum Mechanics, bound states are often treated by splitting the Hamilton operator into a part with known spectrum and known wave functions, and a perturbation whose effect can be systematically calculated in a power series in some small parameter. This procedure is successful, because all truly non-perturbative effects are collected in a compact manner in the lowest order wave functions. In Quantum Field Theory, however, the basic quantities are the Green functions, and there is no simple analogue to wave functions. This makes the treatment of bound states inherently more complicated. The salient feature of the effective non-relativistic theory is that it allows for a consistent Hamiltonian formulation in the context of a field theory.

To see how this works, we define the Hamiltonian pertaining to the Lagrangian (3.6),

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_C + \mathcal{H}_I, \\ \mathcal{H}_0 = h^\dagger \left(M - \frac{\triangle}{2M} \right) h, \quad \mathcal{H}_C &= -h^\dagger \frac{\lambda\phi}{2M} h, \quad \mathcal{H}_I = -\sum_{n=1}^{\infty} \frac{1}{(2M)^{2n+1}} \mathcal{L}_{2n+1}. \end{aligned} \tag{3.16}$$

The static external field preserves the time invariance of the theory. We perform canonical quantization at $t = 0$, introducing creation and annihilation operators

$$\mathbf{a}^\dagger(\mathbf{p}) = \int d^3\mathbf{x} e^{i\mathbf{p}\mathbf{x}} h^\dagger(0,\mathbf{x}), \quad \mathbf{a}(\mathbf{p}) = \int d^3\mathbf{x} e^{-i\mathbf{p}\mathbf{x}} h(0,\mathbf{x}) \,, \tag{3.17}$$

obeying the commutation relation $[\mathbf{a}(\mathbf{p}), \mathbf{a}^\dagger(\mathbf{q})] = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})$. The Hamilton operator is defined by $\mathbf{H} = \int d^3x \mathcal{H}(0,\mathbf{x})$ and similarly for $\mathbf{H}_0$, $\mathbf{H}_\mathrm{C}$ and $\mathbf{H}_\mathrm{I}$. A free non-relativistic particle is described by the state $|\mathbf{p}\rangle = \mathbf{a}^\dagger(\mathbf{p})|0\rangle$, which obeys $\mathbf{H}_0|\mathbf{p}\rangle = (M + \mathbf{p}^2/2M)|\mathbf{p}\rangle$. The Coulomb wave functions ψ_n are defined as the solutions of the non-relativistic Schrödinger equation,

$$\left(M + \frac{\mathbf{p}^2}{2M} \right) \psi_n(\mathbf{p}) - \frac{2\pi\lambda}{M} \int d\nu(\mathbf{q}) \frac{1}{|\mathbf{p}-\mathbf{q}|^2} \psi_n(\mathbf{q}) = E_n \psi_n(\mathbf{p}), \tag{3.18}$$

with

$$E_n = M - \frac{M}{2n^2} \left(\frac{\lambda}{2M} \right)^2 \tag{3.19}$$

for the discrete spectrum. Note that the wave functions (2.16) satisfy the equation (3.18).

The eigenstates $(\mathbf{H}_0 + \mathbf{H}_\mathrm{C})|n\rangle = E_n|n\rangle$ of the unperturbed Hamilton operator are given by

$$|n\rangle = \int d\nu(\mathbf{p}) \psi_n(\mathbf{p}) |\mathbf{p}\rangle. \tag{3.20}$$

Our task is to find the isolated poles of the two-point function (3.8). Here we note that the spectrum of the full theory is also encoded in the singularities of the resolvent

$$\mathbf{G}(E) = \frac{1}{E - \mathbf{H}} \,. \tag{3.21}$$

If the Hamilton operator annihilates the vacuum, $\mathbf{H}|0\rangle = 0$, the resolvent is closely related to the two-point function,

$$\langle \mathbf{p}|\mathbf{G}(E)|\mathbf{k}\rangle = -G_{NR}(E;\mathbf{p},\mathbf{k}) \ , \tag{3.22}$$

see Appendix D. The advantage of using the resolvent instead of the Green function is due to the fact that the bound state poles can be isolated in a very convenient manner. As a lowest order approximation, we ignore $\mathbf{H}_{\rm I}$. The resolvent

$$\bar{\mathbf{G}}(E) = \frac{1}{E - \mathbf{H}_0 - \mathbf{H}_{\rm C}} \tag{3.23}$$

of the unperturbed Hamilton operator is proportional to the Coulomb Green function given by Schwinger's solution,

$$\langle \mathbf{p}|\bar{\mathbf{G}}(E)|\mathbf{k}\rangle = (2\pi)^3\, G_S(E-M;\mathbf{p},\mathbf{k})\Big|_{Ze^2=\frac{\lambda}{2M}}. \tag{3.24}$$

If the perturbation is small, the poles of $\mathbf{G}(E)$ are expected to lie in the vicinity of those of $\bar{\mathbf{G}}(E)$. To work out the exact position of the poles, it is convenient to first consider the operator $\mathbf{T}_{\rm I}$ defined through

$$\mathbf{T}_{\rm I}(E) = \mathbf{H}_{\rm I} + \mathbf{H}_{\rm I}\bar{\mathbf{G}}(E)\mathbf{T}_{\rm I}(E) \ . \tag{3.25}$$

It is related to the full resolvent by $\mathbf{G}(E) = \bar{\mathbf{G}}(E) + \bar{\mathbf{G}}(E)\mathbf{T}_{\rm I}(E)\bar{\mathbf{G}}(E)$. Applying a technique introduced a long time ago by Feshbach [17], we use the projector $|n\rangle\langle n|$ to exhibit the pole near the unperturbed value E_n,

$$\mathbf{T}_{\rm I}(E) = \mathbf{T}_n(E) + \frac{\mathbf{T}_n(E)|n\rangle\langle n|\mathbf{T}_n(E)}{E - E_n - \langle n|\mathbf{T}_n(E)|n\rangle}, \tag{3.26}$$

where $\mathbf{T}_n$ is defined by

$$\mathbf{T}_n(E) = \mathbf{H}_{\rm I} + \mathbf{H}_{\rm I}\bar{\mathbf{G}}_n(E)\mathbf{T}_n(E)\,, \qquad \bar{\mathbf{G}}_n(E) = \bar{\mathbf{G}}(E)(\mathbf{1} - |n\rangle\langle n|). \tag{3.27}$$

The corresponding decomposition of the full resolvent is obtained by expressing $\mathbf{G}(E)$ through $\mathbf{T}_n(E)$,

$$\mathbf{G}(E) = \bar{\mathbf{G}}_n(E) + \bar{\mathbf{G}}_n(E)\mathbf{T}_n(E)\bar{\mathbf{G}}_n(E) + \frac{(\mathbf{1} + \bar{\mathbf{G}}_n(E)\mathbf{T}_n(E))|n\rangle\langle n|(\mathbf{1} + \mathbf{T}_n(E)\bar{\mathbf{G}}_n(E))}{E - E_n - \langle n|\mathbf{T}_n(E)|n\rangle}. \tag{3.28}$$

The energy of the bound state is determined by the pole position $\bar{E}_n$ in the full resolvent. Since both $\bar{\mathbf{G}}_n(E)$ and $\mathbf{T}_n(E)$ are regular in the vicinity of E_n that includes also $\bar{E}_n$, the pole position is given by the zero of the denominator in the last term of Eq. (3.28) [17],

$$\bar{E}_n - E_n - \langle n|\mathbf{T}_n(\bar{E}_n)|n\rangle = 0. \tag{3.29}$$

So far, we did not make any approximations or pose restrictions on the perturbation $\mathbf{H}_{\rm I}$. In fact the last formula holds even when we add a "radiation field", see Section VII. Moreover,

the same formula can be used for the analysis of the decays of quasistable bound states [2] - without introducing the concept of "wave function of the unstable state". The problem of determining the positions of the poles in the two-point function $G_{NR}(E;\mathbf{p},\mathbf{k})$ is thus reduced to the calculation of the matrix elements $\langle n|\mathbf{T}_n(E)|n\rangle$ in old-fashioned perturbation theory.

Having taken the non-perturbative part into account, it remains to solve Eq. (3.29) in perturbation theory. In the case at hand, $\lambda/2M$ is the only small parameter and we can express $\bar{E}_n$ as a power series

$$\bar{E}_n = E_n + \sum_{m=3}^{\infty} \lambda^m C_{mn}. \tag{3.30}$$

The coefficients C_{mn} can be obtained by using in (3.29) the iteration

$$\langle n|\mathbf{T}_n(\bar{E}_n)|n\rangle = \langle n|\mathbf{H}_{\rm I}|n\rangle + \langle n|\mathbf{H}_{\rm I}\bar{\mathbf{G}}_n(\bar{E}_n)\mathbf{H}_{\rm I}|n\rangle + \cdots \quad , \tag{3.31}$$

which can be calculated by inserting a complete set of states - corresponding to $\mathbf{H_0} + \mathbf{H_C}$ - and keeping terms of the appropriate order in λ. Here, we recognize the familiar Rayleigh-Schrödinger perturbation theory.

C. Energy-levels

With a given Lagrangian, and a framework set, one can now calculate the energy of the bound-state levels in the non-relativistic theory - the final answer should reproduce of course the relativistic result. For example, for the ground-state energy in the non-relativistic theory one should obtain order by order in perturbation theory in the coupling constant λ

$$E_1 = \Big(M^2 - \frac{\lambda^2}{4}\Big)^{1/2} = M - \frac{\lambda^2}{8M} - \frac{\lambda^4}{128M^3} - \frac{\lambda^6}{1024M^5} + \cdots . \tag{3.32}$$

Using Eq. (3.31), one encounters two conceptual difficulties. First, as was demonstrated in the preceding discussion, the non-relativistic Lagrangians which can be derived from the relativistic theory, are not unique, although all of them yield the same S-matrix elements. In the case considered here, they differ by terms which vanish by the use of EOM and field redefinitions. Second, although all couplings in the non-relativistic Lagrangian are finite, it contains terms with arbitrary high powers of space derivatives. This means that the matrix elements of the perturbation Hamilton operator between the unperturbed wave functions start to diverge at a sufficiently high order of the perturbative expansion. Consequently, one has to demonstrate that: i) the terms which vanish by the use of EOM or which can be removed by field redefinitions, do not contribute to the bound-state energy, and ii) divergences in the bound-state energy cancel when all terms at a given order in the coupling constant are summed up. We shall ensure that this is what indeed occurs in explicit calculations.

In order to evaluate the energy-level shift, one has to specify the explicit form of the Hamiltonian. We find it useful to start with the FWT Hamiltonian, which has a remarkably compact form,

$$\mathcal{H}_{FWT} = h^\dagger \sqrt{M^2 - \triangle - \lambda\phi}\ \ h = h^\dagger\Big(M - \frac{\triangle + \lambda\phi}{2M}\Big)h - h^\dagger\Big(\frac{(\triangle + \lambda\phi)^2}{8M^3} + \cdots\Big)h\,. \tag{3.33}$$

The first two terms in $\mathcal{H}_{FWT} = \mathcal{H}_0 + \mathcal{H}_C + \mathcal{H}_I$ correspond to the unperturbed non-relativistic Coulomb problem, and the rest is considered as a perturbation. In order to regularize the ultraviolet divergences mentioned, we perform the calculations in d space dimensions [$d = D - 1$], and perform the limit $d \to 3$ at the end. We define the Hamilton operator $\mathbf{H}_{\rm FWT} = \int d^d\mathbf{x}\mathcal{H}_{FWT}(0,\mathbf{x})$, and similarly for $\mathbf{H}_0$, $\mathbf{H}_{\rm C}$ and $\mathbf{H}_{\rm I}$, etc. Further, in d dimensions, the momentum-space Coulomb wave function $\psi^d(\mathbf{p})$ is chosen to obey the d-dimensional Schrödinger equation with the same kernel as in Eq. (3.18),

$$\left(E_n^d - M - \frac{\mathbf{p}^2}{2M}\right)\psi_n^d(\mathbf{p}) = -\frac{2\pi\lambda}{M}\int d\nu_d(\mathbf{q})\,\frac{1}{|\mathbf{p}-\mathbf{q}|^2}\,\psi_n^d(\mathbf{q})\,, \tag{3.34}$$

where $d\nu_d(\mathbf{q}) = d^d\mathbf{q}/(2\pi)^d$, and E_n^d denotes the energy of the n-th eigenstate in d dimensions[3]. Using Eq. (3.34), it is seen that the unperturbed Coulomb wave functions are eigenfunctions of $\mathbf{H}_{\rm I}$,

$$\int d^d\mathbf{x}\,h^\dagger\,(\triangle + \lambda\phi)^k h\,|n\rangle_d = (2M(M - E_n^d))^k\,|n\rangle_d\,, \tag{3.35}$$

where $|n\rangle_d$ is constructed with the use of Coulomb wave function $\psi_n^d(\mathbf{p})$, similarly to Eq. (3.20). In this expression, one can safely put $d = 3$. Using then the perturbation series for the energy-levels, one ensures that Eq. (3.32) holds in all orders in the coupling constant λ.

The simplicity of the above derivation is a consequence of the choice of the Hamiltonian in the FWT form (3.33) and should not be misleading. As we have already mentioned, in general one encounters ultraviolet divergences in individual matrix elements in sufficiently high orders of the non-relativistic expansion. To illustrate this fact, we again consider the original Lagrangian (3.6). The difference $\delta\mathcal{H}_I$ between the pertinent Hamiltonian (3.6) and the FWT Hamiltonian (3.33) contains terms at $O(M^{-5})$ and higher,

$$\delta\mathcal{H}_I = -\frac{\lambda}{64M^5}\,h^\dagger(\phi\triangle^2 + \triangle^2\phi - 2\triangle\phi\triangle)h + \frac{\lambda^2}{32M^5}\,h^\dagger(\triangle\phi^2 - \phi\triangle\phi)h + O(M^{-7})\,. \tag{3.36}$$

It is expected that the level shift induced by the perturbation $\delta\mathcal{H}_I$ vanishes to all orders in λ. Consider matrix elements of the form ${}_d\langle n|\int d^d\mathbf{x}\,h^\dagger(\triangle)^k(\lambda\phi)^{l-k}h|m\rangle_d$, with all possible permutations of the operators. Using dimensional arguments, one finds that all such operators will contribute at order $O(\lambda^{2l})$ to the bound-state energy (up to possible logarithms). Therefore, the energy shift induced by $\delta\mathcal{H}_I$ starts at order $\lambda^6 M^{-5}$ - this term is given by the diagonal matrix element of this operator between the unperturbed wave functions. The following matrix element is then e.g. needed,

$$\begin{aligned}\int d^d\mathbf{x}\,{}_d\langle n|\,h^\dagger\,\triangle\phi\triangle\,h\,|m\rangle_d &= \int d\nu_d(\mathbf{p})\,d\nu_d(\mathbf{k})\,\psi_n^d(\mathbf{p})^*\mathbf{p}^2\,\frac{4\pi}{|\mathbf{p}-\mathbf{k}|^2}\,\mathbf{k}^2\psi_m^d(\mathbf{k}) \\ &= -2\pi\lambda^2\psi_n^d(\mathbf{0})^*\psi_m^d(\mathbf{0})\left(L(\mu) + \ln\frac{\lambda^2}{\mu^2}\right) + O(1)\,, \quad d\to 3\,,\end{aligned} \tag{3.37}$$

---

[3] We have checked numerically that the ground state energy depends smoothly on d in the vicinity of physical space dimension $d = 3$. In particular, no bifurcations occur.

where

$$\psi_m^d(\mathbf{0}) = \int d\nu_d(\mathbf{p})\, \psi_m^d(\mathbf{p}), \tag{3.38}$$

and where $L(\mu)$ is given by Eq. (2.47). Eq. (3.37) illustrates the ultraviolet divergences mentioned. Using the bound-state equation in d dimensions (3.34) in the calculation of all matrix elements from (3.36), one finds that all divergent terms containing $L(\mu)$ cancel at order $\lambda^6 M^{-5}$. Moreover, it can be checked that the diagonal matrix element and, therefore, the energy-level shift due to $\delta\mathcal{H}_I$ vanishes at this order,

$${}_d\langle n|\delta\mathbf{H}_{\rm I}|n\rangle_d = O(\lambda^8 M^{-7})\ . \tag{3.39}$$

Consequently, at this order the theories described by the Lagrangian (3.6) and the FWT Lagrangian, yield the same bound-state energy. We do not intend to construct a proof that this holds to any order.

IV. INCLUDING THE DYNAMICAL LIGHT FIELD

In section III, we have set up a non-relativistic framework that allows one to calculate the energy-levels of a particle moving in a given external field. We wish to extend this construction to the model discussed in section II, and to consider radiative corrections to the energy-levels, generated by the Yukawa interaction with the light field. In order to set up a non-relativistic framework for the calculation of the energy shift, one has to specify the Lagrangian which will be used in these calculations, and the rules that are used to obtain observable quantities. Indeed, the perturbative expansion will be carried out not only in the coupling constant, but also in energies and momenta, because one needs to know the amplitude just in the vicinity of the threshold. In such a multiple expansion, the question of ordering arises - one has to specify the relative magnitudes of the expansion parameters. This then determines what terms in the expansion should be grouped together. A particular ordering is called *power counting scheme*. The calculational framework must then ensure that the ordering is not destroyed by loop corrections - this problem goes under the name "validity of power counting".

In this section, we discuss scattering amplitudes at tree level and identify the problems that show up in loop calculations. In order to keep the arguments simple, we put $\lambda = 0$ in this and in the following section.

A. Tree graphs

In a first step, we construct a non-relativistic Lagrangian that reproduces the tree graphs generated by the relativistic Lagrangian Eq. (2.18). We will restrict the discussion to the one heavy particle sector, and first consider scattering matrix elements in the relativistic theory,

$$\langle p, q_{n+1}, \ldots, q_m \,\mathrm{out}\,|k, q_1, \ldots q_n \,\mathrm{in}\rangle = \langle \ldots \mathrm{in}\,|\ldots \mathrm{in}\rangle + i(2\pi)^4\delta^4(P_f - P_i)T(p,k;q_1,\ldots,q_m)\,;$$
$$m \geq 2\,, n \geq 1\ . \tag{4.1}$$

At tree level, the matrix element T depicted in Fig. 4, is a sum of products of free propagators - each term is of the form

$$e^m \prod_{j=1}^{m-1} \frac{1}{M^2 - P_j^2} \,, \tag{4.2}$$

with

$$P_j^0 = k^0 + \sum_{\nu=1}^{m} c_{\nu,j} q_\nu^0 \,,\, \mathbf{P}_j = \mathbf{k} + \sum_{\nu=1}^{m} c_{\nu,j} \mathbf{q}_\nu \,;\, c_{\nu,j} = \pm 1, 0 \,. \tag{4.3}$$

The low-energy expansion is now defined as follows. We decompose each propagator in a singular and a regular part,

$$\frac{1}{M^2 - P_j^2} = \left(\frac{1}{\omega_j - P_j^0} + \frac{1}{\omega_j + P_j^0} \right) \frac{1}{2\omega_j} \,; \qquad \omega_j = (M^2 + \mathbf{P}_j^2)^{1/2} \,, \tag{4.4}$$

write in the first part

$$\omega_j = M + \frac{\mathbf{P}_j^2}{2M} + \triangle\omega_j = \bar{\omega}_j + \triangle\omega_j \quad , \tag{4.5}$$

and note that $\triangle\omega_j$ is much smaller than the difference $\bar{\omega}_j - P_j^0$, because

$$\triangle\omega_j = O(\mathbf{P}_j^4) \quad , \qquad \bar{\omega}_j - P_j^0 = O(q_j^0) \,, \tag{4.6}$$

everywhere in momentum space except the specific configurations of external momenta for which the difference $\bar{\omega}_j - P_j^0$ is of order $\mathbf{P}_j^4$ (these configurations have zero measure in momentum space). Therefore, one may expand in the first term in $\triangle\omega_j$. Furthermore, in the second term and in the factor $1/\omega_j$, one expands in inverse powers of the heavy mass M. In this manner, each of the propagators in each factor in the product (4.2) becomes a series of terms of non-relativistic propagators. The leading term reads

$$\frac{e^m}{(2M)^{m-1}} \prod_{j=1}^{m-1} \frac{1}{\bar{\omega}_j - P_j^0} \quad , \tag{4.7}$$

whereas the next terms contain additional powers $\triangle\omega_j/(\bar{\omega}_j - P_j^0)$. There are also terms that do not contain any propagators - these originate from the second term in Eq. (4.4). Finally, we also expand the quantities $\triangle\omega_j$ in inverse powers of the heavy mass. As an example of this procedure, we consider Compton scattering, which amounts to $m = 2, n = 1$ in (4.1). The intermediate particle in the s-channel graph has four momentum

$$P_1^\mu = (k^0 + q_1^0, \mathbf{k} + \mathbf{q}_1) \quad , \tag{4.8}$$

and the matrix element becomes

$$e^2 T_s = \frac{e^2}{2\omega_1} \left[\frac{1}{\omega_1 - k^0 - q_1^0} + \frac{1}{\omega_1 + k^0 + q_1^0} \right] \quad , \; \omega_1 = (M^2 + \mathbf{P}_1^2)^{1/2} \,. \tag{4.9}$$

Expanding in the manner described above, we obtain an infinite series of terms,

$$T_s = \frac{1}{2M}\frac{1}{\bar{\omega}_1 - k^0 - q_1^0} - \frac{\mathbf{P}_1^2}{4M^3}\frac{1}{\bar{\omega}_1 - k^0 - q_1^0} + \frac{\mathbf{P}_1^4}{16M^4}\frac{1}{(\bar{\omega}_1 - k^0 - q_1^0)^2} + \frac{1}{4M^2} + \cdots \,. \quad (4.10)$$

These contributions have the following representations as Feynman graphs: non-relativistic propagation, vertex correction, mass insertion, contact term, etc. The contribution in the u-channel may be treated in an analogous way. In this manner, one generates a well-defined series of terms, that may be ordered in inverse powers of the heavy mass M.

The next step consists in the construction of a non-relativistic Quantum Field Theory that generates exactly this series for any tree graph in the one heavy particle sector. We start from the observation that the tree graphs are generated by the classical action, evaluated at the solution to the classical EOM. Retaining only the terms that correspond to the sector with one heavy particle, the corresponding non-relativistic Lagrangian is closely related to the one discussed in section III for the motion in the presence of the external field,

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \ell \partial^\mu \ell + h^\dagger \mathcal{D} h \;\;, \qquad \mathcal{D} = D_+ + ed\ell d(1 + eD_-^{-1} d\ell d)^{-1} \,. \quad (4.11)$$

The differential operators $D_\pm$ and d are defined in section III. In order to arrive at a local Lagrangian, one again expands these differential operators in inverse powers of M - the result is given by Eq. (C2) in Appendix C, with $\lambda\phi \to e\ell$. This Lagrangian is analogous to the effective Lagrangian (3.6), except that now, the field ℓ is time dependent. As a result, some of the operators contain time derivatives of the light field ℓ. Higher order time derivatives can be eliminated in a standard manner, see Appendix C. The final result for the effective Lagrangian is of the form

$$\mathcal{L}_{NR} = h^\dagger D_0 h + \frac{1}{2}\partial_\mu \ell \partial^\mu \ell + \sum_{n=1}^{\infty} \frac{1}{(2M)^n}\,\mathcal{L}_n \,,$$

$$\mathcal{L}_1 = e h^\dagger \ell h \,, \qquad \mathcal{L}_2 = 0 \,, \qquad \mathcal{L}_3 = h^\dagger(\triangle^2 + e(\ell\triangle + \triangle\ell) + e^2\ell^2)h \,, \quad \ldots \,. \quad (4.12)$$

The scattering matrix elements are obtained by evaluating - at tree level - the connected Green functions

$$G_{NR}(x, y; z_1, \ldots, z_m) = i\langle 0|T h(x) h^\dagger(y) \ell(z_1) \ldots \ell(z_m)|0\rangle_c \,. \quad (4.13)$$

The perturbation theory is performed as follows. First, one sums up all mass insertions in the external heavy lines. The Fourier transform of G_{NR} then develops poles, and one ends up with a reduction formula for the scattering amplitude that is similar to Eq. (3.8), except that there are also poles corresponding to the light fields. This procedure generates the low-energy expansion of the relativistic scattering amplitude discussed above.

B. Loops

The evaluation of tree diagrams is performed by expanding in inverse powers of the heavy mass M. Each term in the Lagrangian contributes in a definite order - the expansion makes perfect sense. By calculating loops, one again needs an expansion where the contributions from higher dimensional operators are suppressed in a well defined manner - otherwise, the

framework is not useful. This problem has obtained an extensive coverage in the literature, in the context of different physical problems [10,12,14,15]. We first illustrate the problem in the case of the correlator of two heavy fields. In the following section, we show that the above Lagrangian does lead to a consistent framework, provided that the rules to evaluate the Feynman diagrams are properly adapted.

In order to avoid inessential technical complications, we consider here the two-point function at vanishing 3-momentum $\mathbf{p}$ in D dimensions,

$$G_{NR}(\sigma) = i \int d^D x e^{ip^0 x^0} \langle 0|Th(x)h^\dagger(0)|0\rangle = \frac{1}{M - p^0 - \Sigma_{NR}(\sigma)} \ ; \ \sigma = 2(M-p^0)/M \ . \quad (4.14)$$

Here, we have introduced the dimensionless variable σ for later convenience, and use the quantity Σ_{NR} to collect the contributions from the loops, as usual. The dependence of the propagator and of the self-energy part on the variables e^2 and M is suppressed to ease notation. In the following, we concentrate on the evaluation of Σ_{NR} at lowest non-trivial order e^2. Some of the diagrams that contribute at this order are depicted in Fig. 5. The contribution from diagram Fig. 5A reads

$$\Sigma_{NR}^A = \frac{e^2}{4M^2} \frac{1}{i(2\pi)^D} \int \frac{d^D l}{-l^2} \frac{1}{M - p^0 + l^0 + \frac{\mathbf{l}^2}{2M}} \ . \quad (4.15)$$

Without performing any explicit calculation, we can make the following observations. First, the integral is ultraviolet divergent - one has to add counterterms to the original Lagrangian to render the propagator finite. The divergence is not suppressed in the large M expansion - the counterterm required contributes to the mass term. Further, expanding the integrand in powers of σ, it is seen that there will be terms analytic at $\sigma = 0$, as well as a non-analytic term $\sim \sigma \ln \sigma$, reflecting the fact that the propagator generates a branch point at $p^0 = M$. The contributions from the diagrams Fig. 5B + 5C are given by the integral

$$\Sigma_{NR}^{B+C} = -\frac{e^2}{8M^4} \frac{1}{i(2\pi)^D} \int \frac{d^D l}{-l^2} \, \mathbf{l}^2 \, \frac{1}{M - p^0 + l^0 + \frac{\mathbf{l}^2}{2M}} \ . \quad (4.16)$$

Again, the divergence of this integral is not suppressed in the momentum expansion - the mass term is affected also here. Further, there are contributions analytic in σ, as well as a non-analytic term $\sim \sigma^3 \ln \sigma$. We conclude from these two examples that perturbation theory, based on the Lagrangian (4.12) and using standard dimensional regularization, generates a complicated series of terms: in general, all operators in the effective Lagrangian will contribute e.g. to the mass. This feature of the standard expansion is very similar to the problems already discussed in application to different physical problems [10,12–15].

The reason for the breakdown of the counting rules lies in the inconsistency of the procedure: the non-relativistic effective theory we have constructed is supposed to be valid at very small momenta. On the other hand, using dimensional regularization, we have extended the momentum integration to infinity. In the presence of the hard mass scale M, the propagators are no longer homogeneous functions of momenta, and the procedure fails to suppress contributions from higher dimensional operators. The problem is clearly caused by the contributions from large momenta in the integral. Indeed, at very small momenta, the term $\mathbf{l}^2/(2M)$ in the heavy particle propagator may be dropped - then,the integrand is

homogeneous in the momenta, and the contributions from higher dimensional operators are suppressed, being proportional to σ^{D-3} (σ^{D-1}) for the graph depicted in Fig. 5A (Fig. 5B, Fig. 5C). On the other hand, the high energy contributions generate a contribution that is analytic at $\sigma = 0$. One needs a systematic procedure to subtract these unphysical high-momentum contributions from the loop diagrams. The problem has been discussed in the recent literature [6,10–12]. The prescriptions offered reduce to a set of rules which are applied to the integrands, before the integration is carried out. We illustrate it in the following section.

V. THE TWO-POINT FUNCTION

In order to illustrate the method that allows one to subtract the uninteresting polynomial part, we consider here in detail the two-point function of the heavy field at order e^2, in the absence of the external field.

A. The two-point function at p=0

We first consider again the case where the three momentum is set to zero, and start with the expression (4.15). Combining the denominators by use of the Feynman-Schwinger type formula

$$\frac{1}{ab} = \int_0^\infty dy \frac{1}{(a+by)^2}, \tag{5.1}$$

one may perform all integrations, with the result

$$\Sigma^A_{NR} = -\frac{e^2}{16\pi^2 M}\left\{\Phi_1 \ln\sigma + \Phi_2\right\} + O(D-4) \ ,$$
$$\Phi_1 = 1 - \frac{1}{W} \ , \quad \Phi_2 = 2[8\pi^2 \bar{L}_M + \ln 2 - 1 + \frac{1}{W}\ln(1+W)] \ , \quad W = \sqrt{1-\sigma} \ . \tag{5.2}$$

In order to remove the pole in Φ_2, a mass counterterm is needed. In order to keep the position of the pole at $p^0 = M$, the finite part of the counterterm has to be appropriately tuned. In addition, it is explicitly seen that the self-energy develops a logarithmic branch point at $p^0 = M$. This singularity is suppressed, in the sense that it is proportional to σ.

We now compare this result with what one obtains through the multipole expansion [6,10–12]. In the present case, this method amounts to expand the integrand in (4.15) in the three-momentum of the massless particle, according to the counting

$$p^0 - M = O(v^2),\ |\vec{p}| = O(v)\,, \qquad l^0 = O(v^2),\ |\vec{l}| = O(v^2)\,, \tag{5.3}$$

where v is a small parameter. Proceeding in this manner, an infinite series of terms is generated out of the quantity Σ^A_{NR},

$$\hat{\Sigma}^A_{NR} = \sum_{n=0}^\infty \Sigma_n(\sigma) \ ,$$
$$\Sigma_n = (-1)^n \frac{e^2}{4M^2}\frac{1}{i(2\pi)^D}\int \frac{d^D l}{-l^2}\left(\frac{\mathbf{l}^2}{2M}\right)^n \frac{1}{(M - p^0 + l^0)^{n+1}} \ . \tag{5.4}$$

The hat on $\hat{\Sigma}^A_{NR}$ indicates that the result of this manipulation is not identical to the previous expression Σ^A_{NR}. It is seen that Σ_n is of order $v^{2D+2n-6}$ near threshold. Let us consider the first term in the series,

$$\Sigma_0 = \frac{e^2}{4M^2}\frac{\Gamma(D-2)\Gamma(3-D)}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}(M-p^0)^{D-3} \ . \tag{5.5}$$

In contrast to the case considered before, this term is now suppressed near threshold, being of order v^2 at $D \to 4$. The mass is unaffected by the loop correction, and the counterterm needed to cancel the ultraviolet divergence is also of order v^2, provided that we count the differential operator $i\partial_t - M + \frac{\triangle}{2M}$ as a quantity of order v^2. Furthermore, the non-analytic part in this first term,

$$\Sigma_0 = \frac{e^2}{32\pi^2 M}\sigma \ln\sigma + \cdots \ , \tag{5.6}$$

exactly agrees with the non-analytic part in the original expression (5.2). This feature is true to all orders in the v^2 expansion (5.4). Indeed, the integrations can be done in each term, by integrating first over l^0 in the lower half plane. In the limit $D \to 4$, the summation can be explicitly performed, with the result

$$\begin{aligned}\hat{\Sigma}^A_{NR} &= -\frac{e^2}{16\pi^2 M}\left\{\Omega_1 \ln\sigma + \Omega_2\right\} \ , \\ \Omega_1 = \Phi_1, &\qquad \Omega_2 = \left(1-\frac{1}{W}\right)\left\{8\pi^2\bar{L}_M + \ln 2 - 1 - \ln(1+W)\right\} \ .\end{aligned} \tag{5.7}$$

It is seen that indeed, the logarithmic part in Σ^A_{NR} is exactly reproduced, whereas the second contribution Φ_2, analytic at threshold, is modified, as a result of which Σ_n is of the order expected from counting powers of v in the integrand. [The summation of the series in Eq. (5.4) has been carried out for illustrative purpose only - in order to demonstrate that the infrared-singular part of the multipole-expanded self-energy diagram indeed reproduces the exact result given by Eq. (5.2) in all orders in the expansion in σ. In practice such a summation is never used - instead, only a finite number of terms is retained that contribute to the matching condition in a given accuracy.]

The self-energy (5.7) has, aside from the branch point at $p^0 = M$, in addition a branch point at $p^0 = M/2$ on the first sheet in the complex p^0 plane. This additional branch point, however, lies outside the region $p^0 \sim M$ where both the non-relativistic approach and the multipole expansion are supposed to be meaningful. The close analogy with the "infrared regularization" of Feynman integrals in Baryon ChPT [15] is evident. Further, it is seen from Eq. (5.7) that the ultraviolet-divergent term is no longer polynomial in momenta after the summation - the renormalization procedure can be carried out only order by order in the v^2 expansion.

We conclude that in this example, the two expressions Σ^A_{NR} and $\hat{\Sigma}^A_{NR}$ differ by a power series in σ. Next, we investigate the contributions from the graphs Fig. 5B + 5C. Proceeding analogously, it is seen that these start to contribute now at order σ^{D-1}. We have not carried out the summation of the multipole expanded terms in this case. However, we have checked that the leading term in the logarithmic singularity again agrees with the one in the original expression Σ^{B+C}_{NR} in Eq. (4.16).

We see that the multipole expansion eliminates the unwanted terms that do not respect power counting. In addition, it is clear from the procedure that contributions generated by the higher order terms in the effective Lagrangian will be suppressed in the momentum expansion in an analogous manner. At each order in the expansion, the ultraviolet divergences may be removed by adding a local term to the effective Lagrangian. These terms again are suppressed by inverse powers of M, as a result of which one has a perfectly consistent procedure. The self-energy $\hat{\Sigma}_{NR}$ becomes in the multipole expansion

$$\hat{\Sigma}_{NR} = \sum_{n=1}^{\infty} \sigma^n \left\{ c_n \ln \sigma + d_n \right\} + O(D-4) \ . \tag{5.8}$$

The terms of order σ^n are generated by a *finite* number of diagrams. Further, the branch point in the propagator stays at M at each order in this expansion, there is no need to adjust the mass term in the Lagrangian for this to achieve.

B. The two-point function at $\mathbf{p} \neq 0$.

So far, we have considered the propagator at zero three momentum. Let us now consider the general case,

$$G_{NR}(p) = i \int d^D x e^{ipx} \langle 0 | T h(x) h^\dagger(0) | 0 \rangle \ . \tag{5.9}$$

As before, we consider the corresponding self-energy at order e^2. The complication that arises for non-vanishing three momentum is the following. The contributions from the mass insertions $h^\dagger (\Delta/M^2)^n h$ generate singular contributions near $p^0 = M + \mathbf{p}^2/(2M)$ for small three momenta (see Fig. 6), of the form

$$\text{const} \times e^2 \mathbf{p}^{2n} / (M + \frac{\mathbf{p}^2}{2M} - p^0)^m \ . \tag{5.10}$$

The origin of this singularity becomes very clear in the multipole expansion. Indeed, summing up the most singular parts of these divergences, one finds that the self-energy behaves as

$$\hat{\Sigma}_{NR}(z, \mathbf{p}^2) = \frac{e^2}{16\pi^2 M} z \ln z + \cdots \quad , \qquad z = (\omega_{\mathbf{p}} - p^0)/M \quad , \tag{5.11}$$

near the threshold $z = 0$. Whereas the lowest order diagram Fig. 5A generates a branch point at $p^0 = M + \mathbf{p}^2/(2M)$, the mass insertions move this to $p^0 = \omega_{\mathbf{p}}$, as it is required from the fact that the theory should describe the non-relativistic limit of the relativistic theory [21]. This is analogous to the free propagator, that has, in the absence of any interaction, a pole at $p^0 = M + \mathbf{p}^2/2M$ - only after an infinite re-summation of mass insertions does this change to the required form $1/(\omega_{\mathbf{p}} - p^0)$.

The calculation of the non-relativistic propagator may now be performed as follows. All calculations are done using the multipole expansion, counting $M + \mathbf{p}^2/(2M) - p^0$ and $\mathbf{p}^2$ as quantities of order v^2. At each order in this expansion, the counterterms required may be chosen such that $\hat{\Sigma}_{NR}$ remains finite at $D \to 4$,

$$\hat{\Sigma}_{NR}(z,\mathbf{p}^2) = \frac{e^2}{16\pi^2 M}\sum_{n=1}^{\infty} z^n \left\{ c'_n z^{D-4} + d'_n \right\}$$
$$= \frac{e^2}{16\pi^2 M}\sum_{n=1}^{\infty} z^n \left\{ \bar{c}_n \ln z + \bar{d}_n \right\} + O(D-4) \ , \tag{5.12}$$

where the coefficients $c'_n,\ d'_n$ and $\bar{c}_n,\ \bar{d}_n$ are functions of the three-momentum, $c'_n = c'_n(\mathbf{p}^2)$, etc. This procedure generates a well defined series at each order in the expansion, and each term in the series (5.12) is again generated by a finite number of diagrams. The behavior of the two-point function near the pole at $p^0 = \omega_{\mathbf{p}}$ is given by

$$\hat{G}_{NR}(p) \equiv \frac{1}{\omega_{\mathbf{p}} - p^0 - \hat{\Sigma}_{NR}(z,\mathbf{p}^2)} \to \frac{Z_{NR}}{\omega_{\mathbf{p}} - p^0} + O(z^{D-5}) \ , \tag{5.13}$$

with

$$Z_{NR} = 1 + \frac{e^2}{16\pi^2 M^2}\, d'_1 + O(e^4) \ . \tag{5.14}$$

Equation (5.13) may be used to define the mass in the non-relativistic theory - it is the parameter M with the property that the two-point function has a pole at $p^0 = \sqrt{M^2 + \mathbf{p}^2}$. For a detailed discussion of this point, see Ref. [21].

For later use, we give the explicit expression of the multipole-expanded self-energy in the non-relativistic theory at order e^2v^2. At this order, only the graph shown in Fig. 5A contributes. One should add the counterterm that is needed to cancel the ultraviolet divergence - the corresponding Lagrangian is given by Eq. (6.1). The self-energy is given by

$$\hat{\Sigma}_{NR} = \frac{e^2}{4M^2}\ \frac{(Mz)^{D-3}}{(4\pi)^{\frac{D-1}{2}}}\ \frac{\Gamma(D-2)\Gamma(3-D)}{\Gamma(\frac{D-1}{2})} - \frac{e^2 f_1 z}{4M}$$
$$= \frac{e^2 z}{16\pi^2 M}\left\{ \ln 2z + \ln\frac{M}{\mu} - 1 - f_1^r(\mu) \right\} + O(D-4)\,, \tag{5.15}$$

where at $O(v^2)$, $Mz = M + \mathbf{p}^2/(2M) - p^0$, and

$$f_1 = \frac{1}{4\pi^2}\left(\bar{L}(\mu) + f_1^r(\mu)\right) + O(e^2)\,. \tag{5.16}$$

From Eq. (5.15), one can read off the expression for the wave function renormalization factor,

$$Z_{NR} = 1 - \frac{e^2 f_1}{4M^2} + O(e^2v^2, e^4). \tag{5.17}$$

C. Comparison with the relativistic theory

It is instructive to compare at this stage these results with the two-point function of the relativistic theory considered in section II. In the absence of the external field, one has

$$i\int d^D x\, e^{ipx}\,\langle 0|TH(x)H^\dagger(0)|0\rangle = \frac{1}{M^2-p^2-\Sigma(p^2)}. \tag{5.18}$$

The relevant contributions at order e^2 are displayed in Fig. 1A, 1D. Adding the counterterms, we find

$$\Sigma(p^2) = \frac{e^2}{(4\pi)^{D/2}}\,\Gamma(2-\frac{D}{2})\,\int_0^1 dx\,[xM^2-x(1-x)p^2]^{D/2-2} - \delta M^2\ . \tag{5.19}$$

Performing the limit $D\to 4$ before going to the mass-shell, we obtain

$$\Sigma(p^2) = \frac{e^2}{16\pi^2}\,\frac{M^2-p^2}{p^2}\,\ln\frac{M^2-p^2}{M^2} + O(D-4)\,. \tag{5.20}$$

Now, changing the sequence of the limiting procedures and differentiating Σ at $p^2=M^2$ and $D>4$, one obtains the wave function renormalization constant Z,

$$Z = 1 - \frac{e^2}{2M^2}\,\bar{L}_M + \frac{e^2}{16\pi^2 M^2} + O(e^4)\ . \tag{5.21}$$

By comparing the leading logarithmic singularity in Σ with the one in $\hat{\Sigma}^A_{NR}$ given by Eq. (5.11), it is seen that they are identical, up to a factor $1/2\omega_{\mathbf{p}}$. Based on the results in this section, we expect that higher order non-analytic terms in the relativistic self-energy will be reproduced by the multipole expansion as well. The polynomial part can then be adapted by a suitable choice of the counterterms, as a result of which one obtains

$$\hat{\Sigma}_{NR}(z,\mathbf{p}^2) = \frac{1}{2\omega_{\mathbf{p}}}\ \Sigma(p^2)\ \ ,\ p^2\simeq M^2\ \ , \tag{5.22}$$

order by order in the v expansion. We refer the interested reader to Ref. [21] for a discussion of this relation beyond one-loop accuracy.

VI. POWER COUNTING AND MATCHING

We now introduce the external field ϕ again, and consider radiative corrections of order e^2 (one loop) to the scattering matrix elements discussed in section III. We restrict the discussion to Green functions that are relevant for bound state calculations considered later on - i.e., we consider the one heavy particle sector in Fock space, with no incoming or outgoing light particles.

A. The Lagrangian, the reduction formula and power counting

To generate the tree graphs, one replaces the external field $\lambda\phi$ in the Lagrangian given in Eq. (C2), $\lambda\phi\to\lambda\phi+e\ell$, and adds the kinetic term for the light field. Because we do not consider incoming/outgoing light particles, one needs to retain in the interaction Lagrangian only terms that are linear in the field ℓ. In the presence of radiative corrections, as we have already seen, one has to equip this Lagrangian with counterterms that render the theory ultraviolet finite,

$$\mathcal{L}_{NR} = h^\dagger\Big(i\partial_t - M + \frac{\triangle}{2M}\Big)h + \frac{1}{2}\,(\partial_\mu \ell)^2 + \sum_{n=1}^{\infty} \frac{1}{(2M)^n}\,\mathcal{L}_n + \mathcal{L}_{c.t.} \ ,$$

$$\mathcal{L}_1 = h^\dagger(g_1\,\lambda\phi + g_2 e\ell)h \ , \qquad \mathcal{L}_2 = 0 \ ,$$

$$\mathcal{L}_3 = h^\dagger(\triangle^2 + g_3\lambda(\phi\triangle + \triangle\phi) + g_4\lambda^2\phi^2 + g_5 e(\ell\triangle + \triangle\ell) + 2g_6 e\lambda\phi\ell)h \ ,$$

$$\mathcal{L}_{c.t.} = \frac{e^2 f_1}{4M^2}\; h^\dagger\left(i\partial_t - M + \frac{\triangle}{2M}\right)h + \cdots \ . \tag{6.1}$$

The coefficients g_i appearing in the Lagrangian are power series in the variable e^2/M^2, $g_i = 1 + O(e^2)$, and f_1 was determined in (5.16) at leading order in e^2. Scattering matrix elements are calculated by use of the reduction formula Eq. (3.8) except that now - in analogy to the relativistic framework - the external legs of the heavy particles are equipped with wave function renormalization factors $Z_{NR}^{1/2}$, and the radiative corrections to the external legs are omitted. The matrix elements are then expanded in powers of the coupling constants λ and e, and of the external momenta. We define the power counting scheme - that specifies how the momentum expansion is done - by associating the following powers of v to the derivatives in (6.1),

$$i\partial_t - M + \frac{\triangle}{2M} = O(v^2) \ \ , \ \triangle = O(v^2) \ . \tag{6.2}$$

[Note that one may arrange the derivatives in such a manner that the Laplacian never acts on the light field.] In addition, it is understood that the loops are evaluated by use of the multipole expansion considered above: the loop momenta of the light particle are counted as quantities of order v^2, and the propagators of the heavy field are expanded accordingly.

One could now set up a formal power counting for generic graphs containing generic vertices from (6.1). However, we find it more instructive to consider instead two examples, the self-energy discussed in the previous section, and radiative corrections to scattering in the external field at lowest order.

We start the discussion with the self-energy - a few graphs are displayed in Fig. 5. According to (6.2), we count the heavy (light) propagator as a quantity of order $1/v^2$ ($1/v^4$), as a result of which the graph Fig. 5A is counted as a term of order v^2. As we have discussed in the previous section, the multipole expansion transforms the relevant integral into an infinite string of terms, starting at $O(v^2)$. In addition, this graph requires, in the multipole expansion, a string of counterterms that start with the one proportional to f_1 in (6.1). This is indeed a term of order v^2 according to the above counting. The vertices in Fig. 5B, 5C stem from $\mathcal{L}_3$ and generate therefore additional factors of at least the order v^2. The corresponding string of counterterms starts at order v^4, etc. We now turn to scattering in the external field at order λ. Some of the relevant graphs are displayed in Fig. 8. The tree graph Fig. 8A has been evaluated in (3.12). The radiative corrections in Figs. 8B and 8C amount to multiply the tree graph with the residue Z_{NR} of the propagator - this contribution starts at order $e^2\lambda v^0$. The vertex correction Fig. 8D starts at the same order (we discuss below the manner in which one tames the ultraviolet and infrared divergences hidden here). Finally, the graphs Fig. 8E and Fig. 8F contain one vertex from $\mathcal{L}_3$, proportional to g_6. The leading term of this contribution is suppressed by v^2, because it contains one heavy propagator only.

These two examples show two things: First, one associates to any tree or one-loop graph a factor v^m according to the rules spelled out above. The multipole expansion turns the loop contribution into an infinite string of terms, such that the amplitude starts at $O(v^m)$. Second, contributions from higher order operators $\mathcal{L}_n$ are suppressed in the following sense. Consider loop corrections to a matrix element at some fixed order λ^n, and compare the corrections due to a loop generated by $\mathcal{L}_1 \times \mathcal{L}_p$ with the one generated by $\mathcal{L}_1 \times \mathcal{L}_q$, with $p > q$. Then, the leading contribution from the former is suppressed with respect to the one of the latter, and the relevant string of counterterms needed the renormalize the former starts at higher order than the one for the latter: although individual graphs do not contribute at a single order, the starting point of the expansion is hierarchically ordered.

B. Matching: The vertex function at order e^2

In order to determine the coefficients g_i in (6.1), one evaluates the amplitudes at a fixed order in λ, e and requires that the results be the same in the relativistic and in the non-relativistic theory at a given order in v. As the effect of the operators of higher dimensions are suppressed in the multipole expansion, only a finite number of coefficients contributes at a fixed order v. In the following section, we will show that we need only the coefficient g_1 at order e^2 for our purpose. It suffices to evaluate a matrix element that fixes this term, and we choose scattering of the heavy particle in the external field ϕ at order $e^2\lambda$.

The relevant diagrams in the relativistic theory are displayed in Fig. 7 (with counterterms omitted), while those of the non-relativistic scattering matrix T_{NR} are shown in Fig. 8. As we are considering a massless theory, we use dimensional regularization to also regularize the infrared divergences.

The matching condition of relativistic and non-relativistic scattering amplitudes reads

$$T_{NR}(\mathbf{p}, \mathbf{k}) = T(\mathbf{p}, \mathbf{k}) \,. \tag{6.3}$$

At order λ, these scattering amplitudes can be written as

$$\begin{aligned} T(\mathbf{p}, \mathbf{k}) &= \lambda\phi(\mathbf{p} - \mathbf{k})\tilde{T}(\mathbf{p}, \mathbf{k}) \,, \\ T_{NR}(\mathbf{p}, \mathbf{k}) &= \lambda\phi(\mathbf{p} - \mathbf{k})\tilde{T}_{NR}(\mathbf{p}, \mathbf{k}) \,. \end{aligned} \tag{6.4}$$

At $D \neq 4$, both T and T_{NR} can be expanded in the momenta. Since we only need to determine the coefficient g_1, it suffices to work at vanishing momenta, and to require that

$$\tilde{T}_{NR}(\mathbf{0}, \mathbf{0}) = \tilde{T}(\mathbf{0}, \mathbf{0}) \,. \tag{6.5}$$

We start the calculation with the relativistic amplitude. The contribution from diagram Fig. 7E has been evaluated in section II. The diagram Fig. 7D is ultraviolet finite and easy to calculate at zero three-momenta in D dimensions. Adding the contribution from the wave function renormalization constant Z gives

$$\tilde{T}(\mathbf{0}, \mathbf{0}) = 1 + \frac{5e^2}{96\pi^2 M^2} + O(e^4) \,. \tag{6.6}$$

As expected, the infrared divergences that are present in the residue Z and in the vertex function Fig. 7D, cancel at threshold. Indeed, infrared finite cross sections are obtained by adding diagrams from Bremsstrahlung. At threshold, the phase space for these processes vanishes - therefore, the amplitude must be infrared finite itself.

Next we calculate the amplitude in the non-relativistic theory. The vacuum polarization diagram is absent in this case, while the contributions from Fig. 8E, 8F are suppressed in the v expansion. The contribution from diagram Fig. 8D is ultraviolet divergent in the multipole expansion. In order to separate this divergence from the one in the infrared region, we proceed as follows. We first consider the amplitude off-shell,

$$\Gamma_{NR}(p^0) = \frac{e^2}{4M^2 i(2\pi)^D} \int \frac{d^D l}{-l^2} \frac{1}{(M - p^0 + l^0)^2} \,. \tag{6.7}$$

Evaluating this expression in D dimensions gives

$$\Gamma_{NR}(p^0) = \frac{e^2}{M^2} \frac{\Gamma(D-2)\Gamma(4-D)}{4(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})} (M - p^0)^{D-4} \,. \tag{6.8}$$

Now, we eliminate its divergence at $D = 4$ by using

$$g_1 = 1 + \frac{e^2}{16\pi^2 M^2} (\bar{L}(\mu) + g_1^r(\mu)) + O(e^4) \quad , \tag{6.9}$$

where $g_1^r(\mu)$ denotes the finite part in the coupling constant g_1 at order e^2. Finally, keeping $D > 4$, multiplying with Z_{NR} the lowest-order contribution displayed in Fig. 8A and summing up all contributions, we obtain

$$\tilde{T}_{NR}(\mathbf{0}, \mathbf{0}) = 1 + \frac{e^2}{16\pi^2 M^2} (g_1^r(\mu) - f_1^r(\mu)) + O(e^4) \,. \tag{6.10}$$

[Remark: under the following sequence of limiting procedures: going to mass-shell $p^0 \to M$ at $D > 4$ before performing the limit $D \to 4$, the unrenormalized vertex function Γ_{NR} vanishes. It is only the polynomial part proportional to g_1, that survives in this limit.]

The matching condition (6.3) now determines the difference of the finite parts of the counterterms,

$$g_1^r(\mu) - f_1^r(\mu) = \frac{5}{6} + O(e^2) \,. \tag{6.11}$$

C. The Hamiltonian

In order to calculate the energy-levels, we need to construct the Hamiltonian pertaining to the Lagrangian (6.1). As is discussed in Appendix C, we first re-scale the heavy field, such that the kinetic term has coefficient one. As we already mentioned and as is proven in the following section, only the terms proportional to $g_{1,2}$ are needed in the calculation of the energy-level at order $e^2\lambda^2$. Further, terms of order e^2 in g_2 do not contribute. Therefore, re-scaling the heavy field amounts to the renormalization

$$g_1 \to g_1 - \frac{e^2}{4M^2}\, f_1 = 1 + \frac{5e^2}{96\pi^2 M^2} \quad , \tag{6.12}$$

as a result of which the Lagrangian becomes

$$\mathcal{L}_{NR} = h^\dagger\Big(i\partial_t - M + \frac{\triangle}{2M}\Big)h + \frac{1}{2}\,(\partial_\mu \ell)^2 + h^\dagger\Big[\,\frac{e}{2M}\ell + \frac{\lambda}{2M}\Big(1 + \frac{5e^2}{96\pi^2 M^2}\Big)\,\phi\Big]\,h + \cdots \quad . \tag{6.13}$$

The ellipsis denotes terms that are not needed in the following.

Remark: This Lagrangian was obtained without specifying the couplings f_1, g_1 - only the combination (6.12) counts, which is fixed by the matching condition (6.11). The Lagrangian (6.13) amounts to using $f_1 = 0$, which leads to $Z_{NR} = 1$ at this order in the low-energy expansion. As can be seen from the explicit expression (5.15), the derivative of the non-renormalized self-energy indeed vanishes at $D > 4$, which is a particular manifestation of so-called "no-scale" arguments. In this setting, the self-energy is not finite at $D = 4$. However, the pole position of the two-point function is not affected, and S-matrix elements remain the same.[4]

Finally, we display the Hamiltonian. Denoting by Π_ℓ the canonically conjugated momentum of the light field, one has

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_C + \mathcal{H}_I \quad , \tag{6.14}$$

with

$$\mathcal{H}_0 = h^\dagger\Big(M - \frac{\triangle}{2M}\Big)h + \frac{1}{2}\,\Pi_\ell^2 + \frac{1}{2}\,(\vec{\nabla}\ell)^2 \quad ,$$

$$\mathcal{H}_C = -h^\dagger\,\frac{\lambda\phi}{2M}\,h \quad , \quad \mathcal{H}_I = -\frac{e}{2M}\,h^\dagger \ell h - \frac{5e^2\lambda}{192\pi^2 M^3}\,h^\dagger\phi h \;. \tag{6.15}$$

We use in the last section these expressions to calculate the energy-level shift of the ground state.

VII. THE GROUND STATE

The radiative shift of the energy-levels is calculated by use of Eq. (3.31),

$$\Delta E_n = \langle n|\mathbf{H}_{\rm I}|n\rangle + \sum_{m\neq n}\frac{\langle n|\mathbf{H}_{\rm I}|m\rangle\langle m|\mathbf{H}_{\rm I}|n\rangle}{E_n - E_m} + \sum_{m,\gamma}\frac{\langle n|\mathbf{H}_{\rm I}|m,\gamma\rangle\langle m,\gamma|\mathbf{H}_{\rm I}|n\rangle}{E_n - E_m - E_\gamma}$$

$$- \Delta E_n \sum_{m\neq n}\frac{\langle n|\mathbf{H}_{\rm I}|m\rangle\langle m|\mathbf{H}_{\rm I}|n\rangle}{(E_n - E_m)^2} + \cdots \,, \tag{7.1}$$

[4]We are under the impression that many of the contemporary bound-state calculations use implicitly an argument of this kind. On the other hand, we are not aware that the relation of this procedure to standard renormalization theory has been made explicit in the literature.

where γ denotes the light particle which is present in the sum over intermediate states in the Fock space. The Hamilton operator is constructed from the Lagrangian displayed in Eqs. (6.1). The interaction Hamiltonian is no longer diagonal and induces transitions between the Fock space vectors with an arbitrary number of light particles/antiparticles. The unperturbed wave functions in this expansion are the eigenfunctions of $\mathbf{H}_0 + \mathbf{H}_{\rm C}$,

$$(\mathbf{H}_0 + \mathbf{H}_{\rm C})|n\rangle = E_n|n\rangle, \quad (\mathbf{H}_0 + \mathbf{H}_{\rm C})|n,\gamma\rangle = (E_n + E_\gamma)|n,\gamma\rangle\,. \tag{7.2}$$

The Hamilton operator contains an infinite string of operators. Only a few of them, with the low mass dimension, contribute to the energy shift in a given accuracy. For the lowest-order term in (7.1), or for the sums with no light particle in the intermediate state, one can apply power-counting arguments: Since the Coulomb wave functions do not depend on the heavy mass M, and the energy denominators are homogeneous functions of M, one can count the powers of M in the energy shift once one knows the explicit powers of M in the couplings of various operators in $\mathbf{H}_{\rm I}$. Given more powers of M in the denominator, one needs additional powers of λ in the numerator, in order to compensate the extra mass dimensions - that is, only a few first terms in the Hamiltonian can contribute to the energy at $O(e^2\lambda^2)$. The same argument with a slight modification applies also to the sum with one light particle in the intermediate state. Summing up the Coulomb wave functions, one ends up with the Schwinger's Green function, with possible insertions of higher-dimensional operators, folded by the light particle loop. Further, one expands the Schwinger's function in powers of the external field - then, each integral has the form already considered above, in the context of the scattering problem. Using the multipole expansion restores the power counting - the leading term is determined by the naive power-counting rules, and the sub-leading terms are explicitly given. Consequently, for this term in the perturbation expansion one can also apply the same power-counting arguments that were used in the terms with no light field. Therefore, one has to examine only few contributions from the lowest-order operators in $\mathbf{H}_{\rm I}$, in order to obtain the energy-level shift in a given accuracy. In particular, the coefficient g_1 is needed in the accuracy $O(e^2)$, whereas it suffices to determine g_2 at tree level, and all higher-order terms can be ignored. To conclude, in order to evaluate the energy-level shift at $O(e^2\lambda^2)$, it suffices to work with the Hamiltonian given by Eqs. (6.14)-(6.15). For a more detailed discussion of the power counting in the bound-state calculations within the framework of non-relativistic effective theories, we refer the interested reader to Refs. [6,7].

Let us now pass to the calculation of the ground-state energy-level shift. The matrix element $\langle 1|\mathbf{H}_{\rm I}|1\rangle$ receives contribution only from the second term in the interaction Hamilton operator,

$$\langle 1|\mathbf{H}_{\rm I}|1\rangle = -\frac{5e^2}{192\pi^2 M^3}\int d\nu(\mathbf{p})\, d\nu(\mathbf{k})\, \psi_1^*(\mathbf{p})\, \frac{4\pi\lambda}{|\mathbf{p}-\mathbf{k}|^2}\, \psi_1(\mathbf{k}) = -\frac{5e^2\lambda^2}{384\pi^2 M^3}\,. \tag{7.3}$$

The second and fourth terms in the perturbative expansion (7.1) do not contribute to the bound-state energy at $O(e^2\lambda^2)$. In the third term, the summation over the Coulomb wave functions can be carried out explicitly, resulting in Schwinger's Green function [18]. Then, the contribution from this term to the energy-level shift can be written as follows

$$\frac{e^2}{4M^2}\int d\nu_d(\mathbf{p})\, d\nu_d(\mathbf{k})\, \psi_1^d(\mathbf{p})^*\psi_1^d(\mathbf{k}) \int \frac{d\nu_d(\mathbf{q})}{2|\mathbf{q}|}\,(2\pi)^d\, G_S(E_1 - M - |\mathbf{q}|; \mathbf{p}-\mathbf{q}, \mathbf{k}-\mathbf{q})\,. \tag{7.4}$$

An important remark is in order. In Eq. (7.4), one has to use the Schwinger's function in d dimensions, defined through the spectral sum over d-dimensional wave functions (see Appendix B). Further, this function can be written as a sum of zero-Coulomb, one-Coulomb and multi-Coulomb parts. In the first two, the generalization to d dimensions does not cause any problems, while the multi-Coulomb part is ultraviolet convergent and can be considered at $d = 3$.

The zero-Coulomb contribution to the bound state energy is given by

$$\Delta E_{II,0} = \frac{e^2}{4M^2} \int d\nu_d(\mathbf{p})\, |\psi_1^d(\mathbf{p})|^2 \int d\nu_d(\mathbf{q})\, \frac{1}{2|\mathbf{q}|}\, \frac{1}{E_1^d - M - |\mathbf{q}| - \frac{(\mathbf{p}-\mathbf{q})^2}{2M}}\,. \tag{7.5}$$

In order to be consistent with matching in the scattering sector, one has to carry out the multipole expansion in this integral as well. The result at $O(e^2\lambda^2)$ then is

$$\Delta E_{II,0} = -\frac{e^2}{2M^2}\, \bar{L}_M \int d\nu_d(\mathbf{p})\, |\psi_1^d(\mathbf{p})|^2\, \frac{\mathbf{p}^2 + 2M(M - E_1^d)}{2M} - \frac{e^2\lambda^2}{64\pi^2 M^3} \left(\ln \frac{\lambda^2}{M^2} - \frac{3}{2} \right). \tag{7.6}$$

The one-Coulomb contribution to the bound-state energy is calculated analogously,

$$\Delta E_{II,1} = -\frac{\pi e^2 \lambda}{2M^3} \int d\nu_d(\mathbf{p})\, d\nu_d(\mathbf{k})\, \psi_1^d(\mathbf{p})^* \psi_1^d(\mathbf{k})\, \frac{1}{|\mathbf{p}-\mathbf{k}|^2} \times \tag{7.7}$$

$$\times \int d\nu_d(\mathbf{q})\, \frac{1}{2|\mathbf{q}|}\, \frac{1}{(E_1^d - M - |\mathbf{q}| - \frac{(\mathbf{p}-\mathbf{q})^2}{2M})(E_1^d - M - |\mathbf{q}| - \frac{(\mathbf{k}-\mathbf{q})^2}{2M})}\,.$$

After carrying out the multipole expansion, we obtain

$$\Delta E_{II,1} = \frac{e^2}{2M^2}\, \bar{L}_M \int d\nu_d(\mathbf{p})\, |\psi_1^d(\mathbf{p})|^2\, \frac{\mathbf{p}^2 + 2M(M - E_1^d)}{2M} + \frac{e^2\lambda^2}{64\pi^2 M^3} \left(\ln \frac{\lambda^2}{M^2} - 6 + 8\ln 2 \right). \tag{7.8}$$

Note that the divergent terms as well as terms non-analytic in λ, cancel in the sum $\Delta E_{II,0} + \Delta E_{II,1}$.

In the multi-Coulomb part of the energy shift, as it was mentioned above, one can directly perform the limit $d \to 3$, since this contribution is ultraviolet and infrared finite. After re-scaling the integration variables, this piece is written as follows

$$\Delta E_{II,m} = -\frac{8\pi^2 e^2 \lambda^2}{M^3} \int d\nu(\mathbf{u}) d\nu(\mathbf{v}) \frac{1}{(1+\mathbf{u}^2)^2} \frac{1}{(1+\mathbf{v}^2)^2} \times \tag{7.9}$$

$$\times \int d\nu(\mathbf{w}) \frac{1}{|\mathbf{w}|\sqrt{1+|\mathbf{w}|}} \int_0^1 \frac{d\rho \rho^{-(1+|\mathbf{w}|)^{-1/2}}}{F(\rho; |\mathbf{w}|; \mathbf{u}, \mathbf{v})}\, \frac{1}{1 + |\mathbf{w}| + (\mathbf{u} - \frac{\gamma}{2M}\mathbf{w})^2}\, \frac{1}{1 + |\mathbf{w}| + (\mathbf{v} - \frac{\gamma}{2M}\mathbf{w})^2}\,,$$

where

$$F(\rho; |\mathbf{w}|; \mathbf{u}, \mathbf{v}) = (\mathbf{u}-\mathbf{v})^2 \rho + \frac{(1-\rho)^2}{4(1+|\mathbf{w}|)} (1 + |\mathbf{w}| + (\mathbf{u} - \frac{\gamma}{2M}\mathbf{w})^2)(1 + |\mathbf{w}| + (\mathbf{v} - \frac{\gamma}{2M}\mathbf{w})^2)\,, \tag{7.10}$$

and $\gamma = \lambda/2$. Taking the limit $\gamma \to 0$, one identifies this expression with the analogous contribution in the relativistic case, Eqs. (2.41) and (2.42). Finally, collecting all contributions to the radiative energy-level shift, we obtain

$$\Delta E_1 = \langle 1|\mathbf{H}_{\rm I}|1\rangle + \Delta E_{II,0} + \Delta E_{II,1} + \Delta E_{II,m} = -\frac{e^2\lambda^2}{64\pi^2 M^3}\Big(\frac{16}{3} - 8\ln 2 + J\Big) + \cdots , \quad (7.11)$$

which is exactly the result (2.43) obtained by using the relativistic framework.

VIII. SUMMARY AND CONCLUSIONS

i) We have considered in this article the energy-levels of a massive scalar particle, which moves in an external Coulomb field and interacts in addition with a massless scalar particle through a Yukawa interaction. This has allowed us to study in detail the non-relativistic effective Lagrangian treatment [4] of the bound-state problem in Quantum Field Theory.

ii) We first considered the relativistic calculation of the energy levels. The calculation serves as a reference point for the non-relativistic calculation performed later on - with the exception of the non-renormalizable interaction considered at the end of section II. Indeed, that interaction mimics the situation encountered in ChPT. We have shown that - at lowest nontrivial order in the relevant coupling constant g - bound-state observables are ultraviolet finite, once the Green functions in the scattering sector are made finite by an appropriate choice of the counterterms. Although we have not proven it, we have no doubt that this will remain true to all orders on g.

iii) A systematic framework for the perturbative calculation of bound-state observables was then developed in the non-relativistic sector, based on Feshbach's [17] technique. At the end, one recovers the conventional Rayleigh-Schrödinger perturbative expansion of the energy-levels. The method allows one in addition to calculate the position of the pole on the second Riemann sheet, in case that the bound state turns out to be unstable.

iv) Although the external field problem may sound trivial at first, actual calculations in the non-relativistic theory reveal that this is not the case. Indeed, applying the Rayleigh-Schrödinger perturbative expansion, one has to verify that the ultraviolet divergences in the matrix elements of high-dimensional operators cancel, provided that all contributions at a given order in coupling constant are summed up. Stated differently, the non-relativistic expansion does not generate additional ultraviolet divergences, despite the fact that the non-relativistic Lagrangian contains terms with arbitrarily high powers of spatial derivatives.

v) We have explicitly checked in an example that the off-shell ambiguity in the effective theory - that can be traced to the presence of terms in the Lagrangian which vanish upon use of the equation of motion or can be removed by field redefinitions - does not affect the energy-level. [Similar results were recently obtained in the context of NRQCD [22] and in the non-relativistic effective field theory approach to the deuteron [23].]

vi) Next, we considered radiative corrections in the non-relativistic sector in some detail. The construction of the relevant effective Lagrangian is based on the following idea. First, one constructs a Lagrangian that agrees at tree level with the relativistic amplitude to all orders in the low-energy expansion. This guarantees that the correct cut structure is obtained at one-loop level, in the sense that the absorptive parts of the relativistic theory are reproduced. Loops then simply amount to close two of the external lines in the tree graphs - therefore, we expect that the non-relativistic S-matrix elements do have the correct non-analytic structure at one-loop order.

vii) We have investigated the two-point function of the heavy particle at second order in the Yukawa coupling. It was demonstrated that - at zero three-momentum - the multipole expansion of the Feynman integral reproduces the non-analytic parts of the original expression, order by order in the expansion. The polynomial part of the integral - which leads to the breakdown of counting rules - is modified by the multipole expansion. This merely amounts to a change of the renormalization prescription and does not affect physical observables.

viii) In addition, we have investigated the structure of the two-point function in the threshold region, and have shown how the correct pole structure emerges from the multipole expansion. Eq. (5.22) displays the expected relation between the non-relativistic and the relativistic self-energies at one-loop order, but at any order in the v-expansion. That relation presumes that one has properly chosen the counterterms.

ix) We have considered the renormalization procedure for the non-relativistic effective theory, including the subtleties that arise due to the presence of zero-mass particles. In particular, the renormalization of two and three-point functions in the non-relativistic theory was treated in detail. Further, we have demonstrated that - at leading order in the low-energy expansion - a procedure which utilizes no-scale arguments in the dimensional regularization is equivalent to the standard renormalization scheme.

x) Constructing finally the non-relativistic Hamiltonian in the presence of the light field, we have verified that it reproduces - at first nontrivial order in the Yukawa coupling - the relativistic expression for the energy-level of the ground state.

In conclusion, we find that dimensional regularization, together with the multipole expansion [6,10–12] and Feshbach's representation [17] of the Green functions, allows a sound evaluation of energy-levels in the non-relativistic models considered. Furthermore, with the same technique, one can determine the position of the poles on the second Riemann sheet for decaying states [2,3]. No use was made of a representation of the Lagrangian in terms of expanded fields [10,12] in order to perform these calculations.

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APPENDIX A: ONE-COULOMB CONTRIBUTION TO THE ENERGY-LEVEL SHIFT

In order to calculate the one-Coulomb contribution to the energy shift, we first combine the denominators in Eq. (2.39) by using Feynman parameters. Then, one obtains

$$\Gamma(E_1;\mathbf{p},\mathbf{k}) = \frac{1}{16\pi^2}\int_0^1 dx\,\frac{\ln g(x)-\ln f(x)}{g(x)-f(x)}\,,$$
$$g(x) = M^2 + x(1-x)(\mathbf{p}-\mathbf{k})^2\,, \qquad f(x) = x(\gamma^2+\mathbf{p}^2)+(1-x)(\gamma^2+\mathbf{k}^2)\,. \tag{A1}$$

Next, we substitute Eq. (A1) into Eq. (2.38), and re-scale the integration variables according to $\mathbf{p}=\gamma\mathbf{u}$, $\mathbf{k}=\gamma\mathbf{v}$. We obtain

$$F_{II,1} = 8z^2(\ln z^2 - 2\ln 2 - 1)\,I_1 + 8z^2\,I_2 + o(z^2)\,, \tag{A2}$$

where

$$I_1 = \int d\nu(\mathbf{u})d\nu(\mathbf{v})\,\frac{1}{(1+\mathbf{u}^2)^2}\,\frac{1}{(1+\mathbf{v}^2)^2}\,\frac{1}{|\mathbf{u}-\mathbf{v}|^2} = \frac{1}{256\pi^2}\,, \tag{A3}$$

and

$$I_2 = \int d\nu(\mathbf{u})d\nu(\mathbf{v})\,\frac{1}{(1+\mathbf{u}^2)^2}\,\frac{1}{(1+\mathbf{v}^2)^2}\,\frac{1}{|\mathbf{u}-\mathbf{v}|^2}\,\frac{(1+\mathbf{u}^2)\ln(1+\mathbf{u}^2)-(1+\mathbf{v}^2)\ln(1+\mathbf{v}^2)}{\mathbf{u}^2-\mathbf{v}^2}\,. \tag{A4}$$

In order to calculate I_2, it is convenient to use the integral representation

$$\frac{(1+\mathbf{u}^2)\ln(1+\mathbf{u}^2)-(1+\mathbf{v}^2)\ln(1+\mathbf{v}^2)}{\mathbf{u}^2-\mathbf{v}^2} = \int_0^\infty dx\Big(\frac{1}{x+1}-\frac{x}{(x+1+\mathbf{u}^2)(x+1+\mathbf{v}^2)}\Big)\,, \tag{A5}$$

and to introduce the Fourier transform of $|\mathbf{u}-\mathbf{v}|^{-2}$,

$$\frac{1}{|\mathbf{u}-\mathbf{v}|^2} = \frac{1}{4\pi}\int d^3\mathbf{r}\,\frac{e^{i(\mathbf{u}-\mathbf{v})\mathbf{r}}}{r}\,. \tag{A6}$$

After this, all integrations in I_2 can be performed analytically, resulting in

$$I_2 = -\frac{5}{128\pi^2}\Big(\frac{1}{2}-\ln 2\Big)\,. \tag{A7}$$

Substituting Eqs. (A3) and (A7) into Eq. (A2), we arrive at the final result displayed in Eq. (2.40).

APPENDIX B: MULTI-COULOMB CONTRIBUTIONS TO THE ENERGY-LEVEL SHIFT

In this Appendix, we present the calculation of the multi-Coulomb contribution to the energy-level shift. In this calculation, a straightforward use of Schwinger's representation encounters a difficulty, since in the integration region $-\infty < q^0 < \infty$ the variable ν in Eq. (2.14) can become larger than 1. In order to overcome this difficulty, we use the spectral representation for the Schwinger's function in terms of the solutions $|n\rangle_d$ of the non-relativistic Schrödinger equation with the Coulomb potential (3.34). By using this representation, the integration over the variable q^0 can be performed with the use of Cauchy's theorem. The whole contribution from the diagrams of the type II to the self-energy part in D dimensions can be written in the following form

$$\Sigma^{II}(E_1^d;\mathbf{p},\mathbf{k}) = -ie^2\sum_n \int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+i0}\,\frac{\langle \mathbf{p}-\mathbf{q}|n\rangle_d\ {}_d\langle n|\mathbf{k}-\mathbf{q}\rangle}{(E_1^d-q^0)^2-(E_n^d)^2+i0}$$

$$= e^2\sum_n \int d\nu_d(\mathbf{q})\,\frac{\langle \mathbf{p}-\mathbf{q}|n\rangle_d\ {}_d\langle n|\mathbf{k}-\mathbf{q}\rangle}{4E_n^d|\mathbf{q}|}\,\left(\frac{1}{E_n^d-E_1^d+|\mathbf{q}|}+\frac{1}{E_n^d+E_1^d+|\mathbf{q}|}\right), \qquad \text{(B1)}$$

where the sum runs both over the discrete and the continuous spectra of the unperturbed Coulomb Hamilton operator.

The representation (B1) now includes all contributions from the diagrams of the type II, apart from the mass counterterm $-\delta M^2$. To single out from this expression the low-energy contribution which contributes at the leading order in λ, we introduce an additional integration over the auxiliary variable x, using the following identities

$$\frac{1}{E_n^d(E_n^d-E_1^d+|\mathbf{q}|)} = \frac{2\theta(E_1^d-|\mathbf{q}|)}{(E_n^d)^2-(E_1^d-|\mathbf{q}|)^2} - \frac{2}{\pi}\int_0^\infty \frac{dx\,(E_1^d-|\mathbf{q}|)}{(x^2+(E_1^d-|\mathbf{q}|)^2)(x^2+(E_n^d)^2)}\,,$$

$$\frac{1}{E_n^d(E_n^d+E_1^d+|\mathbf{q}|)} = \frac{2}{\pi}\int_0^\infty \frac{dx\,(E_1^d+|\mathbf{q}|)}{(x^2+(E_1^d+|\mathbf{q}|)^2)(x^2+(E_n^d)^2)}\,. \qquad \text{(B2)}$$

If now one substitutes Eq. (B2) into Eq. (B1), one can single out the piece corresponding the multi-Coulomb exchanges. [In this piece, one can safely put $D=4$, $E_1^d=E_1$, etc, since there are no ultraviolet divergences.] As far as this contribution is concerned, it is seen that only the part of this expression which does not involve the additional integration over x (the one coming from the first term in the r.h.s. of the first line of Eq. (B2)), contributes at order $O(\lambda^2)$. Other terms can be proven to contribute at order $O(\lambda^4)$. The reason for the use of the above trick is the following. One can observe from Eq. (B2) that now the variables which serve as an "energy variable" in the Schwinger's representation: $\tilde\epsilon = ((E_1-|\mathbf{q}|)^2-M^2)/(2M)$ for the term without x integration, and $\tilde\epsilon(x) = -(x^2+M^2)/(2M)$ for the terms with the x integration, vary in the regions $-M/2 \le \tilde\epsilon \le -\lambda^2/(8M)$ and $-\infty < \tilde\epsilon(x) \le -M/2$, respectively. The corresponding values of the parameter ν in the Schwinger's representation never exceed 1 and, consequently, the integral over $d\rho$ in Eq. (2.14) always converges.

According to the above discussion, with the use of the Schwinger's representation one can now write down the part of the multi-Coulomb contribution to the heavy particle self-energy graph, which gives the leading-order contribution to the energy shift

$$\Sigma^{II,m}(E_1;\mathbf{p}-\mathbf{q},\mathbf{k}-\mathbf{q}) = e^2\int^{|\mathbf{q}|\leq E_1} d\nu(\mathbf{q})\,\frac{1}{2|\mathbf{q}|}\,\frac{2\pi\lambda^2 I(E_1-|\mathbf{q}|;\mathbf{p}-\mathbf{q},\mathbf{k}-\mathbf{q})}{(M^2-(E_1-|\mathbf{q}|)^2)^{1/2}}\times$$
$$\times\frac{1}{M^2+(\mathbf{p}-\mathbf{q})^2-(E_1-|\mathbf{q}|)^2}\,\frac{1}{M^2+(\mathbf{k}-\mathbf{q})^2-(E_1-|\mathbf{q}|)^2}+\cdots\,, \tag{B3}$$

where the quantity I is given by Eq. (2.14), and the ellipsis stands for terms which generate higher-order contributions in λ.

Substituting Eq. (B3) into the expression of the energy-level shift and re-scaling the integration variables $\mathbf{p}=\gamma\mathbf{u}$, $\mathbf{k}=\gamma\mathbf{v}$, $\mathbf{q}=\gamma^2\mathbf{w}/(2M)$, one arrives at the Eqs. (2.41) and (2.42).

APPENDIX C: USE OF THE EQUATION OF MOTION

1. The effective Lagrangian (3.6)

The original form of the Lagrangian is obtained by expanding the propagator D_-^{-1} and the differential operator d - that appear in the operator $\mathcal{D}$ in (3.2) - in inverse powers of the heavy mass,

$$D_-^{-1} = -\left\{d_t+M+\sqrt{M^2-\Delta}\right\}^{-1} = -\frac{1}{2M}\left\{1-\frac{d_t}{2M}+\frac{1}{4M^2}(d_t^2+\Delta)+\ldots\right\}\,,$$
$$d = \frac{1}{\sqrt{2M}}\left\{1+\frac{\Delta}{4M^2}+\ldots\right\}\;\;;\qquad d_t \doteq i\partial_t - M\,. \tag{C1}$$

The result is

$$\mathcal{L}_{NR} = h^\dagger\mathcal{D}h = h^\dagger D_+ h + \sum_{n=1}^{\infty}\frac{1}{(2M)^n}\mathcal{L}^{(n)}\;\;,$$
$$\mathcal{L}^{(1)} = \lambda h^\dagger\phi h\;\;,\;\mathcal{L}^{(2)}=0\;\;,$$
$$\mathcal{L}^{(3)} = \lambda h^\dagger(\phi\triangle+\triangle\phi)h+\lambda^2 h^\dagger\phi^2 h\;\;,\;\mathcal{L}^{(4)} = -\lambda^2 h^\dagger\phi d_t\phi h\;\;,$$
$$\mathcal{L}^{(5)} = \lambda h^\dagger\left(\triangle\phi\triangle+\frac{5}{2}\phi\triangle^2+\frac{5}{2}\triangle^2\phi\right)h$$
$$+\lambda^2 h^\dagger\left(\phi^2\triangle+\triangle\phi^2+3\phi\triangle\phi+\phi d_t^2\phi\right)h+\lambda^3 h^\dagger\phi^3 h. \tag{C2}$$

The time derivatives of second and higher order may be eliminated in the following manner. First, we write the Lagrangian as

$$\mathcal{L}_{NR} = h^\dagger D h\;\;,\qquad D = d_t + A_0 + A_1 d_t + \sum_{n\geq 2}A_n d_t^n\,. \tag{C3}$$

The operator A_0 contains Laplacians and external fields ϕ, and $A_{1,2,3\ldots}$ contain at least two external fields. The two-point function is obtained by evaluating the classical action

$$S = \int d^4x(\mathcal{L}_{NR}+h^\dagger j+j^\dagger h) \tag{C4}$$

at the solution to the classical equation of motion,

$$Dh = -j \ , \tag{C5}$$

as a result of which

$$S = \int d^4x j^\dagger h \ , h = -D^{-1} j \ . \tag{C6}$$

We eliminate the time derivatives successively from Eq. (C5). First, we consider the quantity

$$\bar{S} = \int d^4x j^\dagger \bar{h} \ , \tag{C7}$$

where

$$\bar{h} = -(\frac{1}{1+A_1} D)^{-1} j = -D^{-1} j - D^{-1} A_1 j \ . \tag{C8}$$

The second term does not contribute to on-shell matrix elements, because on one of the external legs, there is no pole: the source j couples directly to the external fields ϕ present in A_1. We conclude that the quantity $\bar{S}$ contains the same on-shell matrix elements as the original classical action S. Stated otherwise, one may use, instead of (C3), the Lagrangian

$$\mathcal{L}_{NR} = h^\dagger D_1 h \ , \tag{C9}$$

with

$$\begin{aligned} D_1 &= \frac{1}{1+A_1} D \\ &= d_t + B_0 + \sum_{m \geq 2} B_m d_t^m \ ; \ B_m = \frac{1}{1+A_1} A_m \ , \ m = 0, 2, 3, \ldots \ . \end{aligned} \tag{C10}$$

The on-shell matrix elements are obtained by inserting in (C6) the solution of

$$D_1 h = -j \ . \tag{C11}$$

By acting with $B_2 d_t$ on (C11), solving for $B_2 d_t^2$ and proceeding as before, one may eliminate $B_2 d_t^2$, and similarly for all higher time derivatives.

2. The effective Lagrangian (4.12)

The main observation is the fact that, as far as tree graphs in the one heavy particle sector are concerned, we may perform in the classical action (C4) the replacement $\lambda\phi \to e D_0^{-1} f$, where f is the external field associated with the light particle, and $\Box D_0^{-1}(x) = \delta^4(x)$. Therefore, the problem of eliminating higher order time derivatives is reduced to the one the previous discussion, except that now, the external field is time-dependent. The time derivatives act on the heavy and on the external field. The ones acting on the heavy field may be removed as is described above. As for the time derivatives on the external field, one uses $\Box D_0^{-1}(x) = \delta^4(x)$ and can thus eliminate all of them, except those of first order.

3. Re-scaling the heavy field

Constructing the Hamiltonian, it is useful to rely on a Lagrangian whose kinetic term is properly normalized. This is not the case for the one displayed in (6.1) due to the presence of the counterterms. Since these are needed only at tree level, one may again use the above described manipulations to eliminate them. However, some of the unwanted time derivatives differ from the ones in (C3), to the extent that the operators A_n need now not contain external fields - an example is the counterterm proportional to f_1 in (6.1). As a result of this, the manipulations described after Eq. (C3) lead to Green functions that differ from the original ones also on the mass-shell - to quote an example, the residue of the two-point function is be modified. However, one can convince oneself that, in the calculation of the S-matrix elements, this change in the two-point function is exactly compensated by a corresponding change of the coupling constants in the effective Lagrangian. As a result of this, the manipulations leading from (C3) to (C10) generate a Lagrangian that leaves the S-matrix elements untouched.

APPENDIX D: CONNECTION OF THE RESOLVENT WITH THE TWO-POINT FUNCTION

In this Appendix we derive Eq. (3.22) of section III, which gives the connection of the resolvent

$$\langle \mathbf{p}| \frac{1}{E-\mathbf{H}} |\mathbf{k}\rangle \tag{D1}$$

with the two-point function

$$G_{NR}(E;\mathbf{p},\mathbf{k}) = i \int dt d^3\mathbf{x} d^3\mathbf{y} e^{iEt - i\mathbf{p}\mathbf{x} + i\mathbf{k}\mathbf{y}} \langle 0|T h(t,\mathbf{x}) h^\dagger(0,\mathbf{y})|0\rangle. \tag{D2}$$

Expressing the creation and annihilation operators of one-particle states in terms of the fields, Eq. (3.17), and using the representation

$$\frac{1}{E-\mathbf{H}} = \frac{1}{i}\int_0^\infty dt e^{i(E-\mathbf{H})t}, \tag{D3}$$

which is valid for Im $E > 0$, as well as the fact that $\mathbf{H}$ generates translations in time

$$h(t,\mathbf{x}) = e^{i\mathbf{H}t} h(0,\mathbf{x}) e^{-i\mathbf{H}t}, \tag{D4}$$

we get

$$\langle \mathbf{p}| \frac{1}{E-\mathbf{H}} |\mathbf{k}\rangle = -\int_0^\infty dt \int d^3\mathbf{x} d^3\mathbf{y} e^{iEt - i\mathbf{p}\mathbf{x} + i\mathbf{k}\mathbf{y}} i\langle 0| e^{-i\mathbf{H}t} h(t,\mathbf{x}) h^\dagger(0,\mathbf{y})|0\rangle. \tag{D5}$$

We assume that the full Hamilton operator annihilates the vacuum $\mathbf{H}|0\rangle = 0$. Since $t > 0$, we may replace the expectation value of the ordinary product with that of the time-ordered product of fields without changing anything. Finally, observing that $\langle 0|h^\dagger(0,\mathbf{y}) = 0$, we can extend the t-integration to $-\infty$ and find

$$\langle \mathbf{p}| \frac{1}{E-\mathbf{H}} |\mathbf{k}\rangle = -G_{NR}(E;\mathbf{p},\mathbf{k}). \tag{D6}$$

Note that the last step is a consequence of the non-relativistic nature of the theory: the number of particles associated with the field h cannot change.

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FIGURE CAPTIONS

FIG. 1. The ultraviolet divergent diagrams in the relativistic theory at one loop. The heavy (light) particle is denoted by a solid (dashed) line, and the wavy line represents the external field. (A) Tadpole diagram, (B) the interaction of the light field with the external field through a heavy loop, (C) the light particle self-energy graph, (D) the heavy particle self-energy graph.

FIG. 2. The self-energy diagrams of the heavy particle moving in an external field (relativistic theory, order e^2). The counterterm contributions are not not shown. The double line corresponds to the propagator $\bar{G}(x, y)$, and the dashed line denotes a light particle.

FIG. 3. Scattering of the heavy particle in the external field at order λ^2. The heavy particle is denoted by a solid line, and the wavy line represents the external field. Graph (A) contributes to the Green function in the relativistic theory, whereas (B) and (C) contribute in the non-relativistic formulation. Crosses denote mass insertions.

FIG. 4. The scattering process $k + q_1 + \cdots + q_n \to p + q_{n+1} + \cdots + q_m$ in the relativistic theory at tree level. The heavy (light) particle is denoted by a solid (dashed) line.

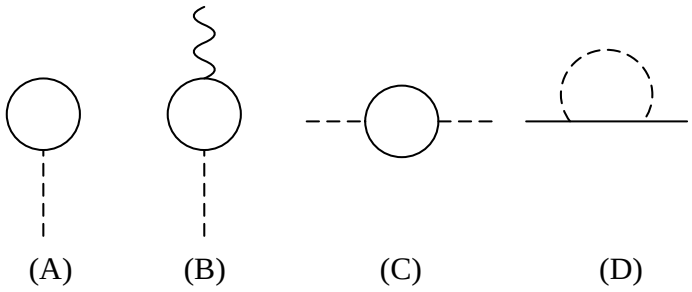
FIG. 5. The self-energy graph of the heavy particle at order e^2, non-relativistic theory. The heavy (light) particle is denoted by a solid (dashed) line. (A) The lowest-order graph with non-derivative vertices, (B) and (C) are graphs with one derivative vertex, denoted by a filled circle.

FIG. 6. Mass insertions (indicated by crosses) in the self-energy of the heavy particle, non-relativistic theory. The heavy (light) particle is denoted by a solid (dashed) line.

FIG. 7. Scattering of the heavy particle in the external field at order $e^2\lambda$, relativistic theory. The heavy (light) particle is denoted by a solid (dashed) line, and the wavy line represents the external field. Diagrams containing counterterms are not shown.

FIG. 8. Scattering of the heavy particle in the external field at order $e^2\lambda$, non-relativistic theory. The heavy (light) particle is denoted by a solid (dashed) line, and the wavy line represents the external field. Diagrams containing counterterms are not shown.

FIGURES



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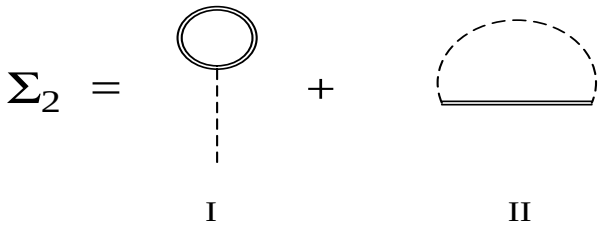

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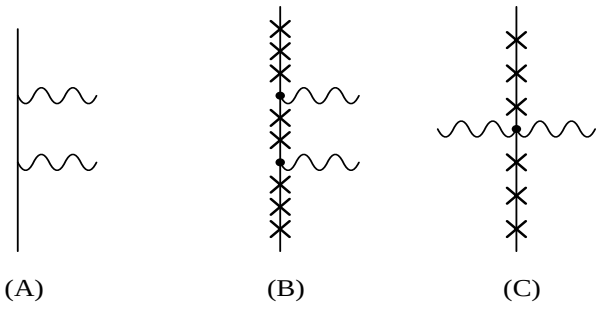

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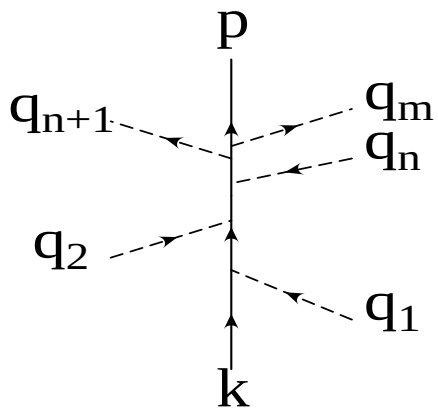

FIG. 4.

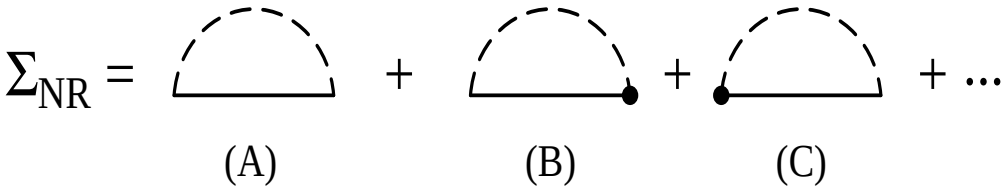

FIG. 5.

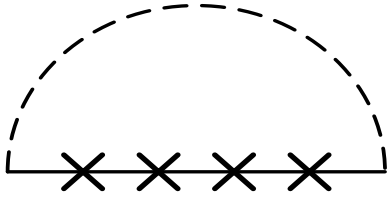

FIG. 6.

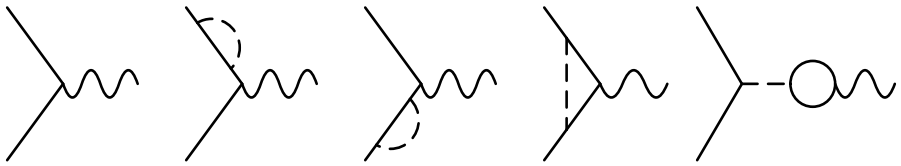

FIG. 7.

FIG. 8.